First Semester Algebra Exam **August 20, 2021**

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let G be a cyclic group. Prove that every subgroup of G is cyclic. (Be sure to consider infinite cyclic groups as well as finite ones.)

2. Let G, H be finite groups with orders m, n.

   (a) Prove that if m and n are coprime then

   $$\mathrm{Aut}(G \times H) \cong \mathrm{Aut}(G) \times \mathrm{Aut}(H).$$

   (b) Give an example where m and n are not relatively prime and

   $$\mathrm{Aut}(G \times H) \ncong \mathrm{Aut}(G) \times \mathrm{Aut}(H).$$

   Justify your claims.

3. (a) Let G be a finite group of order n and let p be the smallest prime which divides n. Prove that if H is a subgroup of G of index p then H is normal in G.

   (b) Give an example of a finite group G and a subgroup $H \leq G$ such that H is not normal in G and the index $|G : H|$ is prime.

4. Let p, q, r be primes such that $p < q < r$ and let G be a group of order pqr. Prove that G has a normal Sylow subgroup.

5. Prove that the alternating group A_5 is simple. Prove that A_4 is solvable.

6. Show that there are four different homomorphisms $\phi : Z_2 \to \mathrm{Aut}(Z_8)$, and prove that the corresponding semidirect products $Z_8 \rtimes_\phi Z_2$ are pairwise nonisomorphic. (Hint: Consider the number of elements of order 2.)