PhD Analysis Exam, August 2018

Do six of eight. Answer each problem on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

(1) Let X, Y be topological spaces. Prove, if $f : X \to Y$ is continuous, then f is Borel measurable.

(2) Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space and let $f \in L^1(\mu)$. Prove that for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $E \in \mathscr{M}$ and $\mu(E) < \delta$, then

$$\int_E |f|\, d\mu < \epsilon.$$

(3) Do two. In each case, give a brief justification for your answer.

(a) Give an example, if possible, of a sequence $(f_n) \subset L^1(\mathbb{R})$ and an $f \in L^1(\mathbb{R})$ such that $f_n \to f$ uniformly on $\mathbb{R}$ but f_n does not converge to f in the L^1 norm.

(b) Evaluate

$$\lim_{n\to\infty} \int_1^\infty \frac{e^{inx}}{x^n}\, dx.$$

(c) Give an example to show that the σ-finiteness hypothesis is necessary in Tonelli's theorem.

(4) Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space and $\mathscr{N} \subset \mathscr{M}$ a sub-σ-algebra. Put $\nu = \mu|_{\mathscr{N}}$. Prove, if ν is σ-finite and $f \in L^1(\mu)$, then there exists a $g \in L^1(\nu)$ such that for all $E \in \mathscr{N}$,

$$\int_E f\, d\mu = \int_E g\, d\nu.$$

(5) Let $(X, \mathscr{M}, \mu)$ be a σ-finite measure space. Fix a measurable function $f : X \to \mathbb{C}$.

(a) Assuming $f \in L^\infty(\mu)$, show if $g \in L^1(\mu)$, then $fg \in L^1(\mu)$ and the mapping $M_f : L^1(\mu) \to L^1(\mu)$ defined by $M_f g = fg$ is a bounded linear transformation.

(b) Conversely, show, if $fg \in L^1(\mu)$ for each $g \in L^1(\mu)$, then $f \in L^\infty(\mu)$.

(6) Fix $1 < p < \infty$ and let $\frac{1}{p} + \frac{1}{q} = 1$. Let (f_n) be a sequence in $L^p[0,1]$. Suppose that for every $g \in L^q[0,1]$, the limit

$$\lim_{n\to\infty} \int_0^1 f_n g$$

exists. Prove that there exists an $f \in L^p[0,1]$ such that

$$\lim_{n\to\infty} \int_0^1 f_n g = \int_0^1 fg$$

for all $g \in L^q[0,1]$. (Hint: first show that the assignment $g \to \lim \int f_n g$ defines a linear functional on L^q.)

(7) Let X be a Banach space. Say a sequence (x_n) from X converges *weakly* to $x \in X$ if $f(x_n) \to f(x)$ for every $f \in X^*$. Prove that if Y is a *closed* subspace of X, $(x_n) \subset Y$, and $x_n \to x$ weakly, then $x \in Y$.

(8) Let X be a Banach space, $Y \subset X$ a closed subspace. Say Y is *complemented* in X if there exists a *closed* subspace $Z \subset X$ such that $Y \cap Z = \{0\}$ and $Y + Z = X$.

Prove, if Y is complemented in X, then there exists a bounded linear operator $T : X \to Y$ such that $T(y) = y$ for all $y \in Y$.