Numerical Linear Algebra Exam-January 2024
Do **4** (four) problems

1. Assume $A \in C^{m\times m}$.
(a) Show that A has a Schur decomposition.
(b) If A has a collection of m linearly dependent eigenvectors, show that A is diagonalizable.

2. Let matrix $A \in \mathcal{C}^{m\times n}$ with $n < m$. Let $b \in \mathcal{C}^m$, and let r denote the residual vector $r = b - Ax$.
(a) Show that x solves the least squared problem $\min \|b - Ax\|_2$ if and only if $r \in Null(A^*)$.
(b) Suppose A is full rank, and describe how to find the least squares solution using the QR decomposition of A.

3. Let $A \in C^{m\times n}$, with $m \geq n$ and $rank(A) = p = n \geq 3$. Let $a_1, a_2, \cdots$ denote the columns of A.
(a) Using the modified Gramm-Schmidt process, write out expressions for q_1, q_2, q_3, the first three columns of Q in the QR decomposition of A.
(b) Show the vector q_3 found in part (a) is orthogonal to both q_1 and q_2.

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.
(a) If E is a $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1 - \|E\|}.$$

(b) If A is a $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

5. If $q_1, \cdots, q_n$ is an orthonormal basis for the subspace $V \subset C^{m\times n}$ with $m > n$, prove that the orthogonal projector onto V is QQ^*, where Q is the matrix whose columns are the q_j.