Numerical Linear Algebra Exam-Summer 2023
Do **4** (four) problems

1. Suppose A is Hermitian positive definite.
(a) Prove that each principal submatrix of A is Hermitian positive definite.
(b) Prove that an element of A with largest magnitude lies on the diagonal.
(c) Prove that A has a Cholesky decomposition.

2. Let $P \in C^{m\times n}$ be a projector. Show that $\|P\|_2 = 1$ if and only if P is an orthogonal projector.

3. Let $A \in C^{m\times n}$, with $m \geq n$ and $rank(A) = p = n \geq 3$. Let $a_1, a_2, \cdots$ denote the columns of A.
(a) Using the modified Gramm-Schmidt process, write out expressions for q_1, q_2, q_3, the first three columns of Q in the QR decomposition of A.
(b) Show the vector q_3 found in part (a) is orthogonal to both q_1 and q_2.

4. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in C^{m\times n}$. Let u_j denote column j of U.
(a) Suppose rank(A)=$p < n < m$. Show $\{u_1, u_2, \cdots, u_p\}$ is a basis for $Col(A)$ and $\{u_{p+1}, u_{p+2}, \cdots, u_m\}$ is a basis for $Null(A^*)$.
(b) Suppose A is full rank and $x \neq 0$. Let σ_i, $i = 1, \cdots, n$ be the sigular values of A. Show

$$\sigma_1 \geq \frac{\|Ax\|_2}{\|x\|_2} \geq \sigma_n > 0.$$

If you want to use the property that $\|A\|_2 = \sigma_1$, then you must prove that.

5. Let $A \in C^{m\times m}$ be Hermetian.
(a) Show that all eigenvalues of A are real.
(b) Define the stationary iterative method (a.k.a. fixed point method)

$$x^{(k+1)} = Ax^{(k)} + b. \qquad (1)$$

Suppose (1) has fixed-point x, namely x satisfies $x = Ax+b$. Show the iteration (1) converges to x from any starting guess $x^{(0)}$, that is $x^{(k)} \to x$ as $k \to \infty$, if and only if the eigenvalues λ_i of A satisfy $|\lambda_i| < 1$, $i = 1, \cdots, m$. You may use the fact that Hermetian matrix A is unitarily diagonalizable.