Numerical Linear Algebra Exam-May 2023
Do **4** (four) problems

1. (a) If P is a projector, prove that $null(P) \cap range(P) = \emptyset$ and $null(P) = range(I - P)$.

(b) Prove that P is an orthogonal projector if and only if it is Hermitian.

2. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.

(a) If E is $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1 - \|E\|}.$$

(b) If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

3. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{pmatrix}.$$

(a) Find a singular value decomposition of A.

(b) Find a QR decomposition of B.

4. Assume $A \in \mathcal{R}^{m \times m}$.

(a) Prove that $\langle x, y \rangle_A = x^* A y$ is an inner product on $\mathcal{R}^m$ if and only if A is symmetric and positive definite.

(b) Assume that A is symmetric and positive definite. If x_* is the solution to $Ax = b$ and $\{p_1, \cdots, p_m\}$ is an orthonormal basis for $\mathcal{R}^m$ with respect to $\langle , \rangle_A$ and $x_* = \sum c_i p_i$ give a formula for the c_i.

5. Let matrix $A \in \mathcal{C}^{m \times n}$ with $n < m$. Let a vector $b \in \mathcal{C}^m$, and let r denote the residual vector $r = b - Ax$.

(a) Show that x solves the least-squares problem $\min \|b - Ax\|_2$ if and only if $r \in Null(A^*)$.

(b) Suppose A is full rank, and describe how to find the least-squares solution using the QR decomposition of A.