1st Year Exam: Numerical Linear Algebra, May, 2022.
Do **4** (four) problems.

1. Prove or provide a counterexample to each of the following.

(a) If matrix A is normal and triangular, then it is diagonal.

(b) Every matrix has a Schur factorization.

2. **(a)** For $v \in \mathbb{C}^n$, define $f_A(v) := (v^* A v)^{1/2}$. Under what condition(s) on $A \in \mathbb{C}^{n\times n}$ is $f_A(\cdot)$ a norm on $\mathbb{C}^n$? (*Full credit will only be given for a general condition or conditions. An example is not sufficient.*) Prove that $f_A(\cdot)$ is a norm on $\mathbb{C}^n$ under the condition(s) you require.

(b) Assuming the condition(s) you require in **(a)**, given a matrix A, determine the best constants α and β to satisfy the inequality

$$\alpha\|v\|_2 \le f_A(v) \le \beta\|v\|_2.$$

3. Let $A = U\Sigma V^*$ be the singular value decomposition of $A \in \mathbb{C}^{m\times n}$. Let u_j denote column j of U.

(a) Suppose $\operatorname{rank}(A) = p < n < m$. Show $\{u_1, u_2, \dots, u_p\}$, is a basis for $\operatorname{Col}(A)$, and $\{u_{p+1}, u_{p+2}, \dots, u_m\}$ is a basis for $\operatorname{Null}(A^*)$.

(b) Suppose A is full rank and $x \neq 0$. Let $\sigma_i,\ i = 1, \dots, n$, be the singular values of A. Show

$$\sigma_1 \ge \frac{\|Ax\|_2}{\|x\|_2} \ge \sigma_n > 0.$$

If you want to use the property that $\|A\|_2 = \sigma_1$, then you must prove that also.

4. Let $\{a_1, \dots, a_n\}$ be a linearly independent set of vectors. Consider the Gram-Schmidt and modified Gram-Schmidt algorithms for computing an orthonormal basis $\{q_1, \dots, q_n\}$ so that $\operatorname{Span}\{q_1, \dots q_k\} = \operatorname{Span}\{a_1, \dots a_k\}$, for each $k = 1, \dots, n$.

Suppose in computing q_2, an orthogonalization error is committed so that $q_2^* q_1 = \epsilon$.

(a) Use the Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

(b) Use the modified Gram-Schmidt algorithm to compute v_3 so that $q_3 = v_3/\|v_3\|$. What is $v_3^* q_2$?

5. Compute the Cholesky decomposition of the following matrix, or explain why it does not exist.

$$A = \begin{pmatrix} 1 & 1/2 & 2 & 3 \\ 1/2 & 5/16 & 3/2 & 5/2 \\ 2 & 3/2 & 17 & 17 \\ 3 & 5/2 & 17 & 31 \end{pmatrix}.$$