1st Year Exam: Numerical Linear Algebra, January, 2022.
Do **4** (four) problems.

1. Let $A \in \mathbb{C}^{m\times n}$ and $B \in \mathbb{C}^{n\times p}$. Prove or provide a counterexample to the following statements. Justify each nontrivial step.

- **(a)** The Frobenius norm satisfies $\|AB\|_F \le \|A\|_F \|B\|_F$.
- **(b)** $\|A\|_2 \le \|A\|_F$, where $\|A\|_F$ is the Frobenius norm of A.

2. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 2 & 3 \\ 2 & 4 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \\ 2 & 0 & 0 \end{pmatrix}.$$

- **(a)** Find an economy singular value decomposition of A.
- **(b)** Find an economy QR decomposition of B.

3. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.

- **(a)** If E is $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \le \frac{1}{1 - \|E\|}.$$

- **(b)** If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \le \frac{\|A^{-1}\|}{1 - \|A^{-1}\|\|E\|}.$$

4. Let $P \in \mathbb{C}^{m\times m}$ be a projector. Show that $\|P\|_2 = 1$ if and only if P is an orthogonal projector.

5. Suppose A is Hermitian positive definite.

- **(a)** Prove that each principal submatrix of A is Hermitian positive definite.
- **(b)** Prove that an element of A with largest magnitude lies on the diagonal.
- **(c)** Prove that A has a Cholesky decomposition.