Ph.D. Exam: Numerical Analysis, January, 2020.

Do **4** (four) of the **first 5** (1-5) and **4** (four) of the **last 5** problems (6-10).

1. Let $A \in \mathbb{C}^{m\times n}$.

(a) Determine constants α and β such that the following inequality holds for the p and ∞ norms of matrix A, for integers $p \geq 1$. Justify your answer.

$$\alpha\|A\|_\infty \leq \|A\|_p \leq \beta\|A\|_\infty.$$

(b) Prove or give a counterexample: $\|A\|_2 \leq \|A\|_F$, where $\|A\|_F$ is the Frobenius norm of A. If you prove this, make sure to justify each nontrivial step.

2. Suppose A is Hermitian positive definite.

(a) Prove that each principal submatrix of A is Hermitian positive definite.

(b) Prove that an element of A with largest magnitude lies on the diagonal.

(c) Prove that A has a Cholesky decomposition.

3. Define the matrices A and B by

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(a) Find both full and economy singular value decompositions of A.

(b) Find both full and economy QR decompositions of B.

4. Let $\|\cdot\|$ be a subordinate (induced) matrix norm.

(a) If E is $n \times n$ with $\|E\| < 1$, then show $I + E$ is nonsingular and

$$\|(I+E)^{-1}\| \leq \frac{1}{1-\|E\|}.$$

(b) If A is $n \times n$ invertible and E is $n \times n$ with $\|A^{-1}\|\|E\| < 1$, then show $A + E$ is nonsingular and

$$\|(A+E)^{-1}\| \leq \frac{\|A^{-1}\|}{1-\|A^{-1}\|\|E\|}.$$

5. Suppose the linear equation $Ax = b$, with

$$A = \begin{pmatrix} \delta & 1 \\ 1 & 1 \end{pmatrix}, \qquad |\delta| < \epsilon_m/4,$$

is solved on a floating-point system with machine-epsilon ϵ_m, using an LU factorization (no pivoting!) followed by forward and back substitution. (You may assume the operations $(+, -, \div, \times)$, do not incur any additional errors).

(a) If $b = (1, 0)^T$, compute the backward error, $\|\hat{b} - b\|_2/\|b\|_2$, where $\hat{b}$ is the data that satisfies $A\hat{x} = \hat{b}$, and $\hat{x}$ is the computed solution.

(b) Is the result of **(a)** sufficient to draw any conclusions about the backward-stability of the algorithm used to compute $\hat{x}$? Explain.

6. **(a)** Starting with the first two Legendre polynomials over $[-1, 1]$, given by $p_0(x) = 1$, and $p_1(x) = x$, find the next three (monic) Legendre polynomials, $p_2(x), p_3(x)$ and $p_4(x)$.

(b) Find the nodes t_0, t_1, t_2 and weights w_0, w_1, w_2 which define the Gaussian Quadrature formula

$$\int_{-1}^{1} f(t)\, \mathrm{d}\, t \approx w_0 f(t_0) + w_1 f(t_1) + w_2 f(t_2),$$

with degree of exactness $m = 5$. **You should find the nodes exactly, and may leave the weights w_0, w_1, w_2, in integral form**.

7. Consider the fixed point problem $x = f(x)$, where $f(x) = e^{-(3+x)}$.

(a) Assuming all computations are done in exact arithmetic, find the largest open interval in $\mathbb{R}$ where the fixed-point iteration $x_{k+1} = f(x_k)$ is ensured to converge.

(b) Write a Newton iteration for finding the fixed-point.

8. Let $\mathcal{P}_1$ be the space of polynomials of degree at most one. Using the norm $\|u\|_2 = \left(\int_a^b u^2 \,\mathrm{d}\, x\right)^{1/2}$

(a) Find the least-squares approximation to $f(x) = x^3$ in $\mathcal{P}_1$ over $[a, b] = [-1, 1]$.

(b) Find the least-squares approximation to $f(x) = x^4$ in $\mathcal{P}_1$ over $[a, b] = [0, 1]$.

9. **(a)** Find a natural cubic spline B_0 on $-1 \le x \le 1$ that satisfies $v(-1) = 0$, $v(0) = 2$, and $v(1) = 0$.

(b) Find a natural cubic spline on $-1 \le x \le 2$ that satisfies $v(-1) = 0$, $v(0) = 2$, $v(1) = 1/2$ and $v(2) = 0$. You may write the solution as $B = B_0 + B_1$, where B_0 is (or is closely related to) the solution to **(a)**, if you like.

10. Let $f \in C^\infty(a - H, a + H)$, and let $h < H$. Let $x_0 = a - h$, $x_1 = a$ and $x_2 = a + h$.

(a) Find the finite difference approximation to $f''(a)$ based the quadratic interpolant p_2 which satisfies $p_2(x_0) = f(x_0)$, $p_2(x_1) = f(x_1)$ and $p_2(x_2) = f(x_2)$ (you should explicitly show how the difference approximation is derived from the interpolant).

(b) Let $\psi_0(h) = \psi(h)$ be the difference approximation to $f''(a)$ found in part **(a)**. Assume (in exact arithmetic) $\psi(h) \to \psi(0) = f''(a)$ as $h \to 0$, and that $\psi(h)$ has the asympototic expansion

$$\psi(h) = \psi(0) + a_2 h^2 + a_4 h^4 + a_6 h^6 + \ldots.$$

Find the general Richardson extrapolation formula for $\psi_k(h)$ based on $\psi_{k-1}(h)$ and $\psi_{k-1}(h/2)$.