Combinatorics Exam - August 2018

1. Find a generating function for the number of positive integer solutions to $x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 = n$.

2. Tickets numbered $\{1, 2, \ldots, n\}$ are drawn from an urn at random one after the other without replacement. What is the probability that the number r is drawn on the k^{th} drawing?

   Among a population of $n + 1$ people, a rumor is spread at random. One person tells the rumor to a second, who in turn repeats it to a third person, etc. What is the probability that the rumor will be told k times without being repeated to any person?

3. Prove that there exists no simple, bipartite, planar graph with minimum degree at least 4.

4. Let μ be the largest eigenvalue of the adjacency matrix of a graph G and Δ the maximum degree of G. Prove that $\mu \leq \Delta$.

   (Hint: Consider the largest, in absolute value, coordinate of a corresponding eigenvector.)

5. Show that a k-regular graph of girth 4 has at least $2k$ vertices, and a k-regular graph of girth 5 has at least $k^2 + 1$ vertices. Draw a 3-regular graph of girth 4 with $2 \cdot 3 = 6$ vertices and a 3-regular graph of girth 5 with $3^2 + 1 = 10$ vertices.

6. Let f_n be the number of all compositions of the integer n into odd parts. Find the exponential order of the numbers f_n.

7. Recall that a permutation is called *even* if it has an even number of even cycles, and it is called *odd* if it has an odd number of even cycles. Also recall that a derangement is a permutation with no 1-cycles.

   Let A_n be the number of all even derangements of length n, and let B_n be the number of all odd derangements of length n. Find an exact formula for $A_n - B_n$.

8. (a) Prove that if a (v, k, λ) BIBD exists with block set $\mathcal{B}$ on a point set V, then the set $\mathcal{B}' = \{V \setminus B \mid B \in \mathcal{B}\}$ is a BIBD.

   (b) Construct a $(7, 4, 2)$ BIBD.