Topology Ph.D Exam

August 2012

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper

1. Prove that the 3-sphere S^3 is not homeomorphic to the 3-space $\mathbb{R}^3$.

2 . Let $A \subset \mathbb{R}^2$ be an infinite countable subspace.

(a) Can A be connected?

(b) Is $\mathbb{R}^2 \setminus A$ connected?

3. Construct a map $f : T^2 \to S^2$ of degree 3 where $T^2 = S^1 \times S^1$ is a torus.

4. Show that there is no map $S^2 \to T^2$ of degree 3.

5. Let GL_n be the space of all invertible real $n \times n$-matrices. Is GL_n compact? Is it connected?

Answer the following with complete definitions or statements or short proofs.

6. Is the 2-sphere S^2 with k points removed homeomorphic to a topological group for

(a) $k = 1$? (b) $k = 2$? (c) $k = 3$? and (d) $k = 0$?

7. State the Lefschetz Fixed Point Theorem.

8. Compute the Euler characteristic $\chi(\mathbb{C}P^5 \times \mathbb{R}P^5 \times S^5 \times S^2)$.

9. State the Baire Category Theorem

10. Does every function $f : \mathbb{N} \to \mathbb{R}$ admit a continuous extension $\overline{f} : \beta\mathbb{N} \to \mathbb{R}$ to the Stone–Čech compactification?

11. State the Five Lemma.

12.What can you say about the k-th cohomology group of a closed oriented manifold for

(a) $k = n$?

(b) $k = n - 1$?

13. Does there exist a covering space of the figure eight that has a non-trivial abelian fundamental group?

14 Describe all connected subsets of the real line $\mathbb{R}$.

15. State the Contraction Mapping Theorem.