Topology Ph.D Exam

May 2012

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper

1. Does there exist a compact space that contains a discrete infinite subspace?

2 . Let X and Y be connected spaces and X, Y is not the one- point space. Show that $X \times Y \setminus \{z\}$ is connected for any $z \in X \times Y$.

3. Prove that $(0, 1)$ and $[0, 1)$ are not homeomorphic.

4. Show that the 2-sphere S^2 is not a retract of the projective plane, as well as the projective plane is not a retract of S^2.

5. Let $GL(n)$ be the space of all invertible real $n \times n$-matrices. IS GL_n is compact? connected?

Answer the following with complete definitions or statements or short proofs.

6. Explain why every map $S^n \to S^1, n > 1$ is null-homotopic.

7. State the Lefschetz Fixed Point Theorem. Explain that every continuous map $D^n \to D^n$ has a fixed point.

8. State the Mayer-Vietoris sequence for homology.

9. Show that the spaces $S^2 \times S^2$ and $S^2 \vee S^2 \vee S^4$ are not homotopy equivalent.

10. Does every function $f : \mathbb{N} \to \mathbb{R}$ admits a continuous extension $\overline{f} : \beta\mathbb{N} \to \mathbb{R}$ of the Stone–Čech compactification?

11. Formulate the Invariance of Domain Theorem.

12. Let M be a closed 4-dimensional simply connected manifold. What can you say about $H_3(M)$?

13. Does there exist a cover space of the figure eight that has the non-trivial abelian fundamental group?

14 .Let X be a compact Hausdorff topological space. Describe all compact subsets of the space $C(X, \mathbb{R})$ of continuous functions $X \to \mathbb{R}$ in uniform topology.

15. State the Universal Coefficient Theorem for Cohomology