Topology Ph.D. Exam

May 2006

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper

1. Can a compact space contain a discrete infinite subspace?

2. Prove that the product of two connected spaces is connected.

3. Give an example of a space which is path connected but not locally connected.

4. Show that the Klein bottle with a deleted closed disk $K \setminus D$ is not a retract of K.

5. Prove that the group $SO(n)$ is compact in the matrix topology.

Answer the following with complete definitions or statements or short proofs.

6. State the Lefschetz fixed point theorem.

7. Construct a map $f : S^3 \to S^3$ of degree 3.

8. State the Simplicial Approximation theorem.

9. Prove that for a path connected topological group G the Euler characteristic $\chi(G)$ is zero.

10. Does every function $f : \mathbb{N} \to \mathbb{R}$ admit a continuous extension $\bar{f} : \beta\mathbb{N} \to \mathbb{R}$ to the Stone-Čech compactification?

11. What can you say about the k-homology groups of a closed orientable n-manifold for

(a) $k = n$?

(b) $k = n - 1$?

12. Give a definition of the compact-open topology for the space of functions.

13. Let $j : S^1 \subset S^4$ be an embedding. Can $\pi_1(S^4 \setminus j(S^1))$ be trivial? If it is not trivial, can the fundamental group $\pi_1(S^4 \setminus j(S^1))$ be abelian?

14. Give a definition of the wedge of a family of topological spaces. Prove that an infinite wedge of circles is not metrizable.

15. State the *Contraction Mapping Theorem.*