

Topology Ph.D Exam

January 2006

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper

1. Describe all metric spaces whose completions are compact.
2. Describe (with a proof) all connected subsets of the real line \mathbb{R} .
3. Let X be a connected normal space having more than one point. Can X be countable?
4. Show that the 2-torus with a deleted point $T \setminus \{x_0\}$ is not a retract of T .
5. Prove that \mathbb{R}^n is not homeomorphic to \mathbb{R}^m for $n \neq m$.

Answer the following with complete definitions or statements or short proofs.

6. Define the degree of a map. Prove that, given a closed oriented n -dimensional manifold M and an integer number d , there exists a map $f : M \rightarrow S^n$ of degree d .
7. Find a degree of the map $f : T \rightarrow T$, $f(x) = x^k$ where $T = S^1 \times S^1$ is the 2-dimensional torus and $x^k = x \dots x$ is the k th power in the group $S^1 \times S^1$: $x^k = (z_1^k, z_2^k)$ for $x = (z_1, z_2)$, $z_1, z_2 \in S^1 \subset \mathbb{C}$.
8. State the Seifert – van Kampen Theorem.

9. Show that two closed orientable surfaces of different genera are not homeomorphic.

10. State the Universal Coefficient Theorem relating integral homology with integral cohomology.

11. Find the fundamental group and homology groups of the Klein bottle.

12. Formulate the Theorem on Invariance of Domain.

13. Let X be a compact Hausdorff topological space. Describe all compact subsets of the space $C(X, \mathbb{R})$ of continuous functions $X \rightarrow \mathbb{R}$ in uniform topology.

14. Define (or describe) the Stone–Čech compactification βX of a topological space X . Show that $\beta\mathbb{Z}$ is not metrizable (here \mathbb{Z} denotes the set of integer numbers in discrete topology.)

15. Give a definition of CW -space. State the Cellular Approximation Theorem.