# Probability Exam    September, 2009

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**(First)** PLEASE PRINT **(Last)** PLEASE PRINT

1. For each fixed $n$, $X_n^1, X_n^2, \cdots, X_n^n$ are independent random variables and have the same Binomial distribution, that is, $\mathbb{P}(X_n^i = 1) = p_n$ and $\mathbb{P}(X_n^i = 0) = 1 - p_n$ for $i = 1, 2, \cdots, n$. Let $S_n = X_n^1 + X_n^2 + \cdots + X_n^n$. Given the condition that $np_n$ converges to a positive constant $\beta$ as $n$ goes to $\infty$, find the limit distribution of $S_n$ and prove it.

2. Let $X$ and $Y$ be two independent random variables with $\mathbb{E}[Y] = 0$. Show that for $p \geq 1$, we have
$$\mathbb{E}|X + Y|^p \geq \mathbb{E}|X|^p.$$

3. Let $X$ be a positive random variable with a probability density function $f(x)$ that is continuous and $U_n = nX - [nX]$, where $[a]$ means the largest integer which is less than $a$. Find the limit distribution of $U_n$ as $n$ goes to $\infty$.

4. Let $X_1, X_2, \cdots, X_n$ be i.i.d. random variables with probability density function $f(x)$, and $M = \max(X_1, X_2, \cdots, X_n)$. Find the conditional expectation $\mathbb{E}[X_1|M]$.

5. For a continuous function $f(x)$ on $[0, 1]$, define
$$P_n(x) = \sum_{i=0}^{n} f(\frac{k}{n}) \binom{n}{k} \; x^k (1-x)^{n-k}.$$
Show that $P_n(x)$ converges to $f(x)$ on $[0, 1]$ as $n$ goes to infinity.

6. Let $S_n$ be a one-dimensional simple random walk, that is, $S_n = X_1 + \cdots + X_n$ where $X_1 +, \cdots, X_n$ are i.i.d. random variables with $\mathbb{P}(X_1 = +1) = \mathbb{P}(X_1 = -1) = 1/2$, and $S_0 = 0$. and let

$$R_n = 1 + \max_{0 \le k \le n} S_k - \min_{0 \le k \le n} S_k$$

be the number of points visited by time $n$. Prove that $R_n/\sqrt{n}$ converges weakly to some limit (in distribution).

7. Let $B_t$ be standard Brownian motion with $B_0 = 0$ and $\tau = \inf\{t : B_t = a + bt\}$, where $a > 0$ and $b$ are two constants. Prove that

   a). $\mathbb{E}_0[e^{-\lambda\tau}] = \exp\{-ab - a\sqrt{b^2 + 2\lambda}\}$

   b). $\mathbb{P}_0(\tau < \infty) = \exp\{-2ab\}$.

8. Let $\{X_n, n = 0, 1, 2, 3 \cdots\}$ be a Markov chain on a countable irreducible state space $\mathbf{S}$. Let $\phi$ be a nonnegative function with

$$\mathbb{E}[\phi(X_1) | X_0 = x] \le \phi(x)$$

for all but a finite number of $x$. Prove that if $\{x | \phi(x) \le M\}$ is finite for any $M < \infty$, then the Markov chain $\{X_n, n = 0, 1, 2, 3 \cdots\}$ is recurrent.