Probability Exam May, 2008

Name: ______________________________ **UFID:** ____________

(First) PLEASE PRINT **(Last)** PLEASE PRINT

1. Let $\mathcal{F}$ be a sigma field. Two random variables X and Y satisfy

$$\mathbb{E}[Y^2|\mathcal{F}] = X^2, \qquad \mathbb{E}[Y|\mathcal{F}] = X$$

Show that $X = Y$ a.s.

2. Let X and Y be two independent random variables with the same probability distribution $F(x)$. If $(X+Y)/\sqrt{2}$ has the same probability distribution $F(x)$, then

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} \, \mathrm{d}\,\mathrm{y}$$

3. Let $\mathbf{f}_n(x_1, \cdots, x_n)$ be the probability density function of the random vector $(X_1, \cdots, X_n)$ and $\mathbf{g}_n(x_1, \cdots, x_n)$ be the probability density function of the random vector $(Y_1, \cdots, Y_n)$. We define

$$Z_n = \frac{\mathbf{g}_n(X_1, \cdots, X_n)}{\mathbf{f}_n(X_1, \cdots, X_n)},$$

if $\mathbf{f}_n(X_1, \cdots, X_n) > 0$, otherwise $Z_n = 0$. Show that $\{Z_n, n = 1, 2, \cdots\}$ is a supermartingale.

4. Let $X_1, X_2, \cdots$ be a sequence of i.i.d random variables with mean 0 and variance $\sigma^2 < \infty$. If $S_n = X_1 + X_2 + \cdots + X_n$, then

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{|S_n|}{\sqrt{n}}\right] = \sqrt{\frac{2}{\pi}}\sigma.$$

5. Let B_t be a standard Brownian motion and define

$$D_n = \max_{n \le t \le n+1} |B_t - B_n|$$

Prove that D_n/n converges to 0 probability one.

6. Consider square integrable random vectores

$$Y = \Big(y(1), y(2), \cdots, y(d)\Big) \qquad \text{and}$$

$$Y_n = \Big(y_n(1), y_n(2), \cdots, y_n(d)\Big) \qquad n \in \mathbb{N}$$

which satisfy

$$\lim_{n\to\infty} \mathbb{E}\Big(|y_n(j) - y(j)|^2\Big) = 0 \qquad \text{for } \ j = 1, 2, \cdots, d$$

Show that the mean of Y_n converges to the mean of Y, and the covariance matrix of Y_n converges to that of Y.

7. Suppose that $\{X_t\}$ and $\{Y_t\}$ are both standard Brownian motion and are independent. Let $\mathcal{F}_t$ denote the σ-field $\sigma\Big(X_s, Y_s \mid s \in [0,\ t]\Big)$. Prove that

$$Z_t = X_t^2 Y_t \ - \ \int_0^t Y_u \, \mathrm{d}\,\mathrm{u}$$

is an $\mathcal{F}_t$-martingale.

8. Let $\mathcal{F}$ be a sub-sigma field of $\mathcal{G}$ and Z be a nonnegative random variable defined on probability space $(\Omega, \mathcal{G}, \mathbb{P})$ with $\mathbb{E}_{\mathbb{P}}[Z] = 1$. Define

$$\mathbb{Q}(A) = \int_A Z \, \mathrm{d}\,\mathbb{P} \quad \text{ for A} \in \mathcal{G}$$

   a). Show that $\mathbb{Q}$ is a probability measure on $(\Omega, \mathcal{G})$.

   b). For a random variable X, we have

$$\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}] \ = \ \frac{\mathbb{E}_{\mathbb{P}}[ZX|\mathcal{F}]}{\mathbb{E}_{\mathbb{P}}[Z|\mathcal{F}]}$$