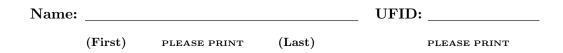
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- 1. Let  $X_1$ ,  $X_2$  be two independent random variables with the same uniform distribution on  $(\theta 1/2, \theta + 1/2)$ , and let  $Y_1 = \min(X_1, X_2), Y_2 = \max(X_1, X_2),$ 
  - (a). Find  $\mathbb{P}(Y_1 \leq \theta \leq Y_2)$ ,
  - (b). Find  $\mathbb{P}(Y_1 \le \theta \le Y_2 | Y_2 Y_1 \ge 1/2)$ .
- 2. Let  $\phi(t)$  be a characteristic function, prove that
  - (a).  $1 Re(\phi(2t)) \le 4(1 Re(\phi(t))),$
  - (b).  $1 |\phi(2t)|^2 \le 4(1 |\phi(t)|^2).$
- 3. Let  $\{X_n, n \ge 1\}$  be a sequence of i.i.d. nonnegative random variables. let  $S_0 = 0$ , and  $S_n = X_1 + \cdots + X_n$ . For t > 0, we define

$$\{\omega | N_t(\omega) = n\} = \{\omega | S_n(\omega) \le t < S_{n+1}(\omega)\}$$

show that

$$\mathbb{E}(N_t(\omega)) = \sum_{n=1}^{\infty} \mathbb{P}(S_n(\omega) \le t).$$

4. Let  $X_1, X_2, \cdots$  be a sequence of strictly positive random variables such that

$$\mathbb{E}(X_{n+1}|\mathcal{F}_n) = f_n(X_n).$$

For  $n \geq 2$ , let

$$M_n = \frac{X_1 X_2 \cdots X_n}{f_1(X_1) f_2(X_2) \cdots f_{n-1}(X_{n-1})}$$

- (a). Show that for  $n \geq 2$ ,  $M_n$  is a  $\mathcal{F}_n$ -martingale.
- (b). Does this martingale converges almost surely and in  $L^{1?}$  Explain it.
- 5. Let  $\{M_n, n \ge 0\}$  be a sequence of integrable random variables adapted to a filtration  $\mathcal{F}_n$ . Assume that for each bounded stopping time T,  $\mathbb{E}(M_T) = \mathbb{E}(M_0)$ , show that  $\{M_n, n \ge 0\}$  is a martingale.
- 6. Let X and Y be random variables such that  $\mathbb{E}(X^2) < \infty$  and  $\mathbb{E}(Y^2) < \infty$ ,  $\mathbb{E}(X|Y) = Y$  and  $\mathbb{E}(Y|X) = X$ . Show that X = Y a.s.

7. Let X, Y be two independent random variables with  $\mathbb{E}[Y] = 0$ . Show that for  $p \ge 1$ 

$$\mathbb{E}[|X|^p) \le \mathbb{E}[|X+Y|^p].$$

8. Let (X, Y) be a random point on a unit circle with uniform distribution, that is

$$\mathbb{P}((X,Y) \in A) = \frac{\operatorname{length}(A)}{2\pi}$$

for any Borel subset A of  $C_2 = \{(x, y) | x^2 + y^2 = 1\}$ . Find the marginal distribution of X.

9. Let  $\mathcal{F}_n$  be a filtration,  $|X_n| \leq Y$ , Y integrable. Suppose that  $X_n \longrightarrow X$  a.e. Using the martingale convergence theorem to prove that

$$\mathbb{E}[X_n|\mathcal{F}_n] \longrightarrow \mathbb{E}[X|\mathcal{F}_\infty] \qquad a.e.$$

10. Let  $Y \in L^p$ ,  $|X_n| \leq Y$  and  $X_n \longrightarrow X$  in distribution. Show that  $X_n$  converges to X in  $L^p$ .