Probability Exam   Spring, 2005

Name: ______________________________ **UFID:** ____________

(First) PLEASE PRINT **(Last)** PLEASE PRINT

1. Let $X_1,\ X_2$ be two independent random variables with the same uniform distribution on $(\theta - 1/2, \theta + 1/2)$, and let $Y_1 = \min(X_1, X_2)$, $Y_2 = \max(X_1, X_2)$,

   (a). Find $\mathbb{P}(Y_1 \le \theta \le Y_2)$,

   (b). Find $\mathbb{P}(Y_1 \le \theta \le Y_2 | Y_2 - Y_1 \ge 1/2)$.

2. Let $\phi(t)$ be a characteristic function, prove that

   (a). $1 - Re(\phi(2t)) \le 4(1 - Re(\phi(t)))$,

   (b). $1 - |\phi(2t)|^2 \le 4(1 - |\phi(t)|^2)$.

3. Let $\{X_n, n \ge 1\}$ be a sequence of i.i.d. nonnegative random variables. let $S_0 = 0$, and $S_n = X_1 + \cdots + X_n$. For $t > 0$, we define

$$\{\omega | N_t(\omega) = n\} = \{\omega | S_n(\omega) \le t < S_{n+1}(\omega)\}$$

   show that

$$\mathbb{E}(N_t(\omega)) = \sum_{n=1}^{\infty} \mathbb{P}(S_n(\omega) \le t).$$

4. Let $X_1,\ X_2, \cdots$ be a sequence of strictly positive random variables such that

$$\mathbb{E}(X_{n+1} | \mathcal{F}_n) = f_n(X_n).$$

   For $n \ge 2$, let

$$M_n = \frac{X_1 X_2 \cdot \cdots \cdot X_n}{f_1(X_1) f_2(X_2) \cdots f_{n-1}(X_{n-1})}$$

   (a). Show that for $n \ge 2$, M_n is a $\mathcal{F}_n$- martingale.

   (b). Does this martingale converges almost surely and in L^1? Explain it.

5. Let $\{M_n,\ n \ge 0\}$ be a sequence of integrable random variables adapted to a filtration $\mathcal{F}_n$. Assume that for each bounded stopping time T, $\mathbb{E}(M_T) = \mathbb{E}(M_0)$, show that $\{M_n,\ n \ge 0\}$ is a martingale.

6. Let X and Y be random variables such that $\mathbb{E}(X^2) < \infty$ and $\mathbb{E}(Y^2) < \infty$, $\mathbb{E}(X|Y) = Y$ and $\mathbb{E}(Y|X) = X$. Show that $X = Y$ a.s.

7. Let $X,\ Y$ be two independent random variables with $\mathbb{E}[Y] = 0$. Show that for $p \geq 1$

$$\mathbb{E}[|X|^p) \leq \mathbb{E}[|X+Y|^p].$$

8. Let (X, Y) be a random point on a unit circle with uniform distribution, that is

$$\mathbb{P}((X,Y) \in A) = \frac{\text{length}(A)}{2\pi}$$

for any Borel subset A of $C_2 = \{(x,y)|x^2 + y^2 = 1\}$. Find the marginal distribution of X.

9. Let $\mathcal{F}_n$ be a filtration, $|X_n| \leq Y$, Y integrable. Suppose that $X_n \longrightarrow X$ a.e. Using the martingale convergence theorem to prove that

$$\mathbb{E}[X_n|\mathcal{F}_n] \longrightarrow \mathbb{E}[X|\mathcal{F}_\infty] \qquad a.e.$$

10. Let $Y \in L^p$, $|X_n| \leq Y$ and $X_n \longrightarrow X$ in distribution. Show that X_n converges to X in L^p.