(1)

PhD. Probability. May 2003

WRITE CLEARLY. STATE PRECISELY RESULTS YOU USE. DO NOT OMIT STEPS.

ANSWER EACH QUESTION ON A SEPARATE SHEET.

1. a. Let $(X_n)_{n=1}^{\infty}$ be an i.i.d sequence of real valued variables on $(\Omega, \mathcal{F}, P)$. Let $\mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Put $S_n = X_1 + \dots + X_n$, for $n \geq 1$ and put $S_0 = 0$.

Which of the following sequences are martingales relative to $\mathcal{F}_n$. For those which are not martingales w.r.t. this filtration, what conditions have to be

a) S_n, $n = 0, 1, 2, \ldots$, $E[|X_1|] < \infty$

b) $X_1^2 + \cdots + X_n^2 - n\lambda$, $n = 1, 2, \ldots$, $E(X_1^2) < \infty$,

λ real

c) $\exp(\alpha S_n - n\lambda)$, $n = 0, 1, 2, \ldots$ where

$\varphi(\alpha) = E[\exp \alpha X_1] < \infty$, α, λ real.

d) $Y_n = S_{\min(n, T)}$, $n = 0, 1, 2 \ldots$

where

$P(X_1 = \pm 1) = \frac{1}{2}$ and $T = \min(n > 0, S_n = 0)$.

$\boxed{2}$ Let $\{\alpha_n\}_{n=1,2,\ldots}$ be a sequence of probability measures on $\mathbb{R}$. Show the following are equivalent:

a) There exists a probability measure α on $\mathbb{R}$ such that

$$\lim_{n \to \infty} \alpha_n(I) = \alpha(I)$$

for all closed intervals I on $\mathbb{R}$ whose end points are continuity points of α.

b) If $\{F_n\}_{n=1,2,\dots}$, respectively F are the distribution functions of $\{\alpha_n\}$, respectively α, .

$$\lim F_n(x) = F(x)$$

at every point x of continuity of F.

$\boxed{3}$ Let $\{\alpha_n\}_{n=1,2,\dots}$ be a sequence of probability measures on $\mathbb{R}$ and let $\{\phi_n(t)\}_{n=1,2\dots}$ be the corresponding characteristic functions. Assume that $\lim_n \phi_n(t) = \phi(t)$ exists $\forall t \in \mathbb{R}$ and $\phi(t)$ is continuous at $t=0$. Prove that $\phi(t)$ is the characteristic function of some probability measure and that the conditions of Problem $\boxed{2}$ hold.

[4] If X is a real valued random variable, show that for $0 < \varepsilon \le 1$,

$$E\left[\frac{|X|}{1+|X|}\right] \le \varepsilon + P\left[|X| > \varepsilon\right]$$

and that

$$P(|X| > \varepsilon) \le \frac{1+\varepsilon}{\varepsilon} E\left[\frac{|X|}{1+|X|}\right].$$

Deduce that the set of real valued random variables on a given probability space can (with suitable identification) be regarded as a metric space if the distance between two random variables X and Y is defined to be

$$d(X, Y) = E\left[\frac{|X-Y|}{1+|X-Y|}\right].$$

and that convergence in this metric space then corresponds to convergence in probability.

[5] a) Let X_n, $n=1,2,\dots$ be i.i.d >0, $E[X_n]=1$ and $Y_n = \prod_1^n X_k$. Prove that Y_n is a martingale relative to the filtration $\mathcal{F}_n = \sigma(X_1,\dots,X_n)$.

Show that $\lim Y_n = Y_\infty$ exists a.s.

b) Show that

$$P[Y_\infty = 0] = 0 \text{ or } 1.$$

Hint: Recall infinite products and the zero-one law.

c) (Using b)) Show that

$$E[Y_\infty] = 1 \iff P[Y_\infty > 0] = 1.$$

[6] Let X_n, $n=1,2,\dots$ be a supermartingale such that $E[|X_{n+1}-X_n| \mid \mathcal{F}_n] \leq M$ a.s.

Show that for any two optional times $S \leq T$, $E(T) < \infty$, we have $E[|X_T|] < \infty$ and

$$E(X_T \mid \mathcal{F}_S) \leq X_S.$$