Ph.D. Probability Exam
May 21, 1998

1. Let X and Y be independent and uniform on $[0,1]$.
   Find
   a) the distribution of $X+Y$
   b) the conditional density of X given $X+Y=z$.

2. Let $X_0, X_1, \ldots, X_n, \ldots$ be i.i.d with mean m. Let N be Poisson with mean λ, independent of the X's. Define

$$Y = X_0 + \cdots + X_N.$$

   Prove that Y is integrable. Find $E(Y)$.

3. State precisely (all for i.i.d)
   a) the strong law of large numbers and prove it.
   b) the central limit theorem.
   c) the law of the iterated logarithm.

4. Define a martingale relative to a filtration $\{\mathcal{F}_n\}$, $n=0,1,2,\ldots$. Define a stopping time relative to this filtration. State the optional sampling theorem. State and prove the martingale convergence theorem.

5. What is an infinitely divisible distribution or $\mathbb{R}^1$. Prove that a non–trivial infinitely divisible distribution cannot have compact support.

6. Define a standard Brownian Motion W starting at 0.
   Define new processes W_1, W_2, W_3 by:

$$W_1(t) = cW(\frac{t}{c^2}),\ t \geq 0,\ c \text{ real } \neq 0$$

$$W_2(t) = tW(\frac{1}{t}),\ t > 0,\ W_2(0) = 0$$

$$W_3(b) = \begin{cases} W(1) - W(1-t), & 0 \leq t \leq 1 \\ W(t) & \text{otherwise.} \end{cases}$$

   Show that W_1, W_2, W_3 are the standard Brownian motions.

7. Let $\varphi(t) = \int e^{itx}\mu(dx)$ be a characteristic function where μ is a probability measure on $\mathbb{R}^1$.
   a) Suppose $|\varphi(t)| = 1$ for some $t \neq 0$. Then prove that unless $\varphi(t) \equiv 1$, there is a smallest $t_0 \neq 0$ and a d such that μ is concentrated on the set $\{d + \frac{2\pi j}{t_0}\}$, $j = 0, \pm1, \pm2, \ldots$
   b) Using a) prove that $|\cos t|$ is not a characteristic function. Hint: $\cos t$ and $\cos^2 t$ are characteristic functions.