Ph.D. Probability Exam

May 21, 1998

1. Let X and Y be independent and uniform on $[0,1]$.
   Find
a) the distribution of $X+Y$
b) the conditional density of X given $X+Y=z$.

2. Let $X_0, X_1, \ldots, X_n, \ldots$ be i.i.d with mean m. Let N be Poisson with mean λ, independent of the X's. Define

$$Y = X_0 + \cdots + X_N.$$

   Prove that Y is integrable. Find $E(Y)$.

3. State precisely (all for i.i.d)
   a) the strong law of large numbers and prove it.
   b) the central limit theorem.
   c) the law of the iterated logarithm.

4. Define a martingale relative to a filtration $\{\mathcal{F}_n\}$, $n = 0, 1, 2, \ldots$. Define a stopping time relative to this filtration. State the optional sampling theorem. State and prove the martingale convergence theorem.

5. What is an infinitely divisible distribution or $\mathbb{R}^1$. Prove that a non–trivial infinitely divisible distribution cannot have compact support.

6. Define a standard Brownian Motion W starting at 0.
   Define new processes W_1, W_2, W_3 by:

$$W_1(t) = cW(\frac{t}{c^2}),\ t \geq 0,\ c \text{ real } \neq 0$$

$$W_2(t) = tW(\frac{1}{t}),\ t > 0,\ W_2(0) = 0$$

$$W_3(b) = \begin{cases} W(1) - W(1-t), & 0 \leq t \leq 1 \\ W(t) & \text{otherwise.} \end{cases}$$

   Show that W_1, W_2, W_3 are the standard Brownian motions.

7. Let $\varphi(t) = \int e^{itx}\mu(dx)$ be a characteristic function where μ is a probability measure on $\mathbb{R}^1$.
a) Suppose $|\varphi(t)| = 1$ for some $t \neq 0$. Then prove that unless $\varphi(t) \equiv 1$, there is a smallest $t_0 \neq 0$ and a d such that μ is concentrated on the set $\{d + \frac{2\pi j}{t_0}\}$, $j = 0, \pm 1, \pm 2, \ldots$
b) Using a) prove that $|\cos t|$ is not a characteristic function. Hint: $\cos t$ and $\cos^2 t$ are characteristic functions.

6. Let $S_n = (U_n, V_n)$ denote the position after n steps of a random walk on $\mathbb{Z}^2$, starting from $(0,0)$. Suppose that

$$U_{n+1} = U_n \pm 1$$
$$V_{n+1} = V_n \pm 1,$$

where the signs are picked by two independent tosses of a fair coin, independently at each step. Define

$$p_n = P(S_n = (0,0))$$
$$\tau_n = \inf\{m > \tau_{n-1} : S_m = (0,0)\}$$

An easy fact is $P(\tau_n < \infty) = P(\tau_1 < \infty)^n$.

(1) Prove that for any random walk $S_n \in \mathbb{Z}^2$ the following are equivalent:
(i) $P(\tau_n < \infty) = 1$ (ii) $P(S_n = (0,0)$ i.e.$) = 1$.
(iii) $\sum_{n=1}^{\infty} P(S_n = (0,0)) = \infty$.

(2) Find a formula for p_n. $(n > 0)$

(3) Use (1) and Stirling's formula $n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n e^S (S \to 0$ as $n \to \infty)$ to show that the diagonal random walk in $\mathbb{Z}^2$ is recurrent.

7. Let $(P_t)_{t>0}$ be a Markov semigroup. Assume $P_t(x, dy) = p_t(x,y)dy$ with $p_t(x,y) = p_t(y,x)$ and $p_t(x,y) \leq M(t)$, where $M(t)$ is a constant depending on t. If μ is a finite signed measure, show that $\int p_t(x,y)\mu(dx)\mu(dy) \geq 0$.

8. An urn at $t = 0$ contains R_0, (≥ 1) red balls and B_0, (≥ 1) black balls, and 'random sampling with double replacement' is called out as follows:

(i) Choose a ball at random (meaning that each ball in the urn has the same probability to be drawn), and note its color;

(ii) Replace this ball in the urn *together with an extra ball of the same color*;

(iii) Go to (i).

Let M_t be the proportion in red balls in the urn after t such operations, so that $M_0 = \frac{R_0}{R_0+B_0}$.

Show that, with respect to a suitable filtration (**which you should specify**) the sequence $M_0, M_1, \ldots, M_t, \ldots$ is a martingale. Prove that M_t converges as $t \mapsto +\infty$ almost surely to a random variable $M_\infty, 0 \leq M_\infty \leq 1$.

Show that

$$M_0 = EM_1 = EM_2 = \cdots = EM_t = EM_\infty,$$

and that

$$M_0^2 \leq E[M_1^2] \leq \ldots E[M_t^2] \leq E[M_\infty^2],$$

and that

$$P(M_\infty = M_0) < 1.$$

NOTE. You may quote without proof any martingale theorems you need, but should state them carefully.

9. Let X_n be independent and all distributed according to standard normal distribution. Find constants C_n such that

$$\limsup \frac{X_n}{C_n} = 1 \qquad \text{a.s.}$$

Hint. $\int_a^\infty e^{-b^2/2} db \leq \frac{1}{a} e^{-a^2/2}$.
and let $c_n^2 = 2\log(n \log n)$.