Ph.D. exam in Probability Fall 11.07..

1. Let $\{X_n\}$ be a submartingale and T a stopping time such that $E[T] < \infty$. Assume

$$E[\,|X_{n+1} - X_n|\;|\mathcal{F}_n] \le M \quad \text{on } (T > n)$$

for some constant M. Show that X_T is integrable and $E[X_0] \le E[X_T]$.

2. Let $X_n \to X$ in probability. Let f be continuous on R^1. Then $f(X_n) \to f(X)$ in probability. Prove this.

3. Can convergence in a.e. be given by a metric? In otherwords is there a metric ρ such that $\rho(X_n, 0) \to 0$ is equivalent to $X_n \to 0$ a.e. Substantiate.

4. Explain the notion of convergence in distribution. Let $X_n \to X$ in distribution. Does it follow that $X_n \to X$ in probability. Suppose $P[X = a] = 1$. Is it then true that $X_n \to X$ in probability. Substantiate all your answers.

5. Let $Y \in L^p$, $|X_n| \le Y$ and $X_n \to X$ in distribution. Show that $E[|X_n|^p] \to E[|X|^p]$

6. Given that f is a characteristic function and $|f(\tfrac{1}{2})| = 1$ what can you conclude?

7. State and give a sketch of the proof of the strong law of large numbers.

8. Let X_n be i.i.d., $E[X_n] = 0$, $E[X_n^2] = 1$. Can this sequence be <u>complete</u> orthonormal. Substantiate.

9. Let μ be a non-trivial infinitely divisible distribution on $\mathbb{R}^1$. Prove that μ cannot have compact support.