Ph.D. Qualifying Exam in Probability
May 1989

1. Let X_n be independent identically distributed random variables and $S_n = \sum_1^n X_i$.

Suppose $$E[\sup_n |\frac{S_n}{n}|] < \infty.$$

Show that
$$E[|X_1| \operatorname{Log}^+ |X_1|] < \infty.$$

Where
$$\log^+ x = \log x \quad \text{if } x \geq 1$$
$$= 0 \quad \text{otherwise.}$$

<u>Hint</u>: $C_n = \prod_{j=1}^n P[|X_j| \leq j] \to c > 0.$

If $T = \inf \{n: |X_n| > n\}$ then

$$\sum_1^\infty \frac{1}{n} \int_{(T=n)} |X_n| dp = \sum_1^\infty \frac{C_{n-1}}{n} \int_{|X_1| > n} |X_1| dp$$
$$\sum_1^\infty \frac{1}{n} \Sigma \int_{(T=n)} |S_{n-1}| dp < \infty.$$

2. Let X_n be independent Poisson variables. Suppose $\sum_1^\infty X_n < \infty$ a.s. Show that $\sum_1^\infty E[X_n] < \infty$.

<u>Hint</u>. Borel Cantelli.

3. a) Define the notion of conditional expectation. In particular explain the notion $E[X|Y]$ where X and Y are random variables.

b) Suppose X, Y are random variables such that $E(X^2)$,

$E(Y^2) < \infty$

and $E[X|Y] = Y$

$E[Y|X] = X$

Show that $X = Y$ a.s.

4. a) A sequence of characteristic functions on R^1 converging pointwise to a characteristic function does so uniformly on compact sets. Explain why - no need to give complete proofs.

b) Let $b_n \in R$ and f a characteristic function not identically equal to 1. Suppose $f(b_n t)$ converges to a characteristic function. Show that $\lim b_n = b \in R$.

c) Let $\phi(t) = \frac{1}{8}[1 + 7e^{it}]$. Show that ϕ is a characteristic function but that $|\phi|$ is not.

Hint. $|\phi|^2$ is the characteristic function of a measure concentrated on $\{-1,0,1\}$.

If $|\phi|$ is the characteristic function of a measure m. Then $|\phi|^2$ is the characteristic function of m*m.

5. Let X_t be a standard Brownian motion.

(i) Show that (X_t), $\left(X_t^2 - t\right)$ and $e^{uX_t - \frac{1}{2}u^2 t}$ are all martingales.

(ii) Let $T_a(w) = \inf\{t: X_t(w) = a\}$ for $a > 0$. The inf being definded to be $+\infty$ if the set is empty. Using (1) Compute $E[e^{-\lambda T_a}]$ for $\lambda > 0$, and show that $T_a < \infty$ a.s. and $E[T_a] = \infty$.

6. State and sketch the proof of the martingale convergence theorem.

7. Using the martingale convergence theorem prove the following:

Let F_n be an increasing sequence of σ-fields, X_n a sequence of random variables dominated by an integrable random variable Y: $|X_n| \leq Y$. Suppose $X_n \to X$ a.s. Then $E[X_n|F_m \to E[X|F_\infty]$a.s.

Hint. Put the $U_m = \sup_{m \geq n} |X_m - X|$

Then for $m \geq n$

$$E[|X_m - X||F_m] \leq E[U_n|F_m].$$

$$E\{\limsup_m \; E[|X_m - X||F_m]\} \leq E\{E[U_n|F_\infty]\} = E[U_m]$$