PH.D QUALIFICATION EXAMINATION ON PARTIAL DIFFERENTIAL EQUATIONS, 2013

NAME:

1.Let Ω be an open subset of $\mathbb{R}^n$ and let $u \in C^2(\Omega)$ be a harmonic function in Ω. Prove that u is analytic in Ω.

2.Let U be an open and bounded set in $\mathbb{R}^n$ with smooth boundary. Suppose u_1 and u_2 are both smooth solutions to

$$\begin{cases} u_t - \Delta u = 0, & \text{in} \quad U_T := U \times (0,T], \\ u = g, & \text{on} \quad \partial U \times [0,T]. \end{cases}$$

If $u_1(x,T) = u_2(x,T)$ for all $x \in U$, prove that $u_1 \equiv u_2$ within U_T.

3. Suppose u is a smooth solution to

$$u_{tt} - \Delta u = 0, \quad \text{in} \quad \mathbb{R}^n \times (0,\infty)$$

and

$$u \equiv u_t \equiv 0 \quad \text{on} \quad B(x_0,t_0) \times \{t = 0\}.$$

Let

$$C = \{(x,t) | \quad 0 \le t \le t_0, \quad |x - x_0| \le t_0 - t. \quad \}.$$

Prove that $u \equiv 0$ within C.

4. Use characteristics to solve the following equation:

$$\begin{cases} x_1 u_{x_1} + 2x_2 u_{x_2} + u_{x_3} = 3u, & \mathbb{R}^3_+ \\ u(x_1,x_2,0) = g(x_1,x_2) & \partial\mathbb{R}^3_+ \end{cases}$$

where g is a given smooth function.

5.Fix $\alpha > 0$ and let $U = B(0,1)$ (open ball). Show there exists a constant C, depending only on n and α, such that

$$\int_U u^2 dx \le C \int_U |Du|^2 dx,$$

provided

$$|\{x \in U| \quad u(x) = 0\}| \ge \alpha, \quad u \in H^1(U).$$

Date: December 20, 2013.

6. Use Hopf-Lax formula to find a solution of

$$\begin{cases} u_t + \frac{1}{2}|Du|^2 = 0, & \mathbb{R}^n \times (0,\infty) \\ u = -\frac{1}{2}|x|, & \mathbb{R}^n \times \{t=0\}. \end{cases}$$

Is this solution unique in the weak sense?

7.Let U be a bounded subset of $\mathbb{R}^n$, $u \in W^{2,p}(U) \cap W_0^{1,p}(U)$ and $2 \le p < \infty$. Prove that

$$\int_U |Du|^p dx \le C(\int_U |u|^p dx)^{\frac{1}{2}}(\int_U |D^2u|^p dx)^{\frac{1}{2}}$$

for some C independent of u.