PH.D QUALIFICATION EXAM: PARTIAL DIFFERENTIAL EQUATIONS 2010 SEPTEMBER

NAME:

1. Let U be either $\mathbb{R}^n_+$ or the unit ball in $\mathbb{R}^n$ $(n \geq 2)$. Write the expression of the Green's function for $-\Delta$ with respect to the Dirichlet condition on ∂U.

2. Let $U \subset \mathbb{R}^n$ be a bounded open set with smooth boundary ∂U, $A = (a_{ij})_{n \times n}$ be a symmetric, positive definite constant matrix. For the following variational form:

$$I(w) := \frac{1}{2} \int_U \sum_{i,j=1}^{n} a_{ij} \partial_i w \partial_j w dx; \; w \in H^1_g := \{w \in H_1(U); \quad w = g \text{ on } \partial U\}$$

where g is a smooth function on ∂U, prove that the minimizer u exists and is smooth.

3. Let $A = (a_{ij})_{n \times n}$ be a symmetric and positive definite constant matrix. Suppose u satisfies

$$\sum_{i,j=1}^{n} a_{ij} \partial_{ij} u = 0, \qquad \text{in } \mathbb{R}^n$$

and $u \geq -1$ in $\mathbb{R}^n$. Show that $u \equiv$ *constant*.

4. Let $\phi(x) = \frac{1}{n(n-2)\alpha(n)} |x|^{2-n}$ be the fundamental solution for the Laplace operator in $\mathbb{R}^n$ $(n \geq 3)$. Here $\alpha(n)$ is the volume of the unit ball B_1. Suppose $f(x) \in C^2(\mathbb{R}^n$ and satisfies ·

$$|D^j f(x)| \leq C(1 + |x|)^{-2-\epsilon-j}, \quad \forall x \in \mathbb{R}^n, \quad j = 0, 1, 2$$

where ϵ is a positive number. Then

$$u(x) = \int_{\mathbb{R}^n} \phi(x - y) f(y) dy$$

satisfies

$$-\Delta u(x) = f(x) \quad \mathbb{R}^n.$$

5. Let $u \in W^{1,p}(\mathbb{R}^n)$ where $p > n$. Show that

$$\frac{1}{r^n} \int_{B(x,r)} |u(x) - u(y)| dy \leq C(n) \int_{B(x,r)} \frac{|Du(y)|}{|x - y|^{n-1}} dy.$$

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6. Write down an explicit formula for a solution of

$$\begin{cases} u_t - \Delta u + cu = f, & \mathbb{R}^n \times (0, \infty), \\ u = g & \mathbb{R}^n \times \{t = 0\} \end{cases}$$

where $c \in \mathbb{R}$.

7. Let $U \subset \mathbb{R}^n$ be an open, bounded subset of $\mathbb{R}^n$ with smooth boundary. Set $U_T = U \times [0, T]$, $\Gamma_T = \bar{U}_T \setminus U_T$ where $T > 0$. Prove that there exists at most one solution $u \in C^2(\bar{U}_T)$ of

$$\begin{cases} u_{tt} - \Delta u = f, & U_T, \\ u = 0 & \Gamma_T \\ u_t = h, & U \times \{t = 0\} \end{cases}$$

where g, h are smooth functions.