PH.D QUALIFICATION EXAMINATION ON PARTIAL DIFFERENTIAL EQUATIONS, 2010

NAME:

1. Suppose $u \in C_1^2(\mathbb{R}^n \times [0,T]) \cap C(\mathbb{R}^n \times [0,T])$ solves

$$\begin{cases} u_t - \sum_{i,j=1}^n a_{ij}\partial_{ij}u = 0, & \text{in} \quad \mathbb{R}^n \times (0,T] \\ u = g \quad \text{on} \quad \mathbb{R}^n \times \{t=0\}. \end{cases}$$

where $(a_{ij})_{n\times n}$ is a positive definite, symmetric constant matrix, g is a given smooth function. Suppose in addition that

$$u(x,t) \le Ae^{a|x|^2}, \quad x \in \mathbb{R}^n, \quad t \in [0,T]$$

for some $a, A > 0$. Prove

$$\sup_{\mathbb{R}^n \times [0,T]} u = \sup_{\mathbb{R}^n} g.$$

2. Let u solve

$$\begin{cases} u_{tt} - u_{xx} = 0 & \mathbb{R}_+ \times (0,\infty) \\ u = g, \quad u_t = h & \text{on} \quad \mathbb{R}_+ \times \{t=0\}, \\ u = 0 & \text{on} \quad \{x=0\} \times (0,\infty). \end{cases}$$

where g, h are smooth, given functions of x such that $g(0) = h(0) = 0$. Write the explicit expression of u.

3. Let B_1 be the unit ball in $\mathbb{R}^n$ $(n \ge 3)$, $p \in (1, \frac{2n}{n-2})$ and $\alpha \in (0,1)$, show that there exists a C depending only on n, p, α such that

$$\left(\int_{B_1} |u|^p dx\right)^{\frac{1}{p}} \le C(n,p,\alpha)\left(\int_{B_1} |\nabla u|^2\right)^{\frac{1}{2}},$$

for all $u \in H^1(B_1)$ that satisfy the following property:

$$|\{x \in B_1; \quad u(x) = 0\}| \ge \alpha |B_1|.$$

4. Suppose $u \in C^2$ solves

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0,\infty).$$

Fix $x_0 \in \mathbb{R}^n$, $t_0 > 0$ and consider the cone

$$C = \{\ (x,t); \quad 0 \le t \le t_0, \quad |x - x_0| \le t_0 - t \quad \}.$$

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Prove that if $u \equiv u_t \equiv 0$ on $B(x_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ in C.

5. Let $(a_{ij}(x))_{n\times n}$ be smooth, symmetric on B_1 and there exist $0 < \lambda < \Lambda$ such that

$$\lambda|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leq \Lambda|\xi|^2, \quad \forall x \in B_1, \quad \forall \xi \in \mathbb{R}^n.$$

Let $b_i (i = 1, 2.., n)$, c and f be smooth functions on B_1. Show that there exists $\delta > 0$ such that if

$$\|b_i\|_{L^\infty(B_1)} + \|c\|_{L^\infty(B_1)} \leq \delta \quad i = 1, .., n$$

then there exists a smooth function u that solves the Dirichlet boundary problem uniquely:

$$\begin{cases} -\sum_{i,j=1}^{n} \partial_i(a_{ij}(x)\partial_j u) + \sum_{i=1}^{n} b_i(x)\partial_i u(x) + c(x)u(x) = f, \quad B_1, \\ \\ u = 0, \quad \text{on} \quad \partial B_1. \end{cases}$$

6. Let $\{u_m\}$ be a sequence of smooth functions satisfying $u_m(x) \leq C$ and $|\Delta u_m| \leq C$ for some constant C uniformly over B_1, the unit ball. Prove that over $B_{1/2}$, along a subsequence, either u_m converges to $-\infty$ uniformly, or u_m converges to a function u in C^2 norm.

7. Let $\{a_{ij}\}_{n\times n}$ $(n \geq 2)$ be a positive definite, symmetric, constant matrix, $b_1, .., b_n, c$ be smooth functions on B_1, the unit ball in $\mathbb{R}^n$. Show that the minimum of the following variational form

$$I(w) = \int_{B_1} \sum_{i,j=1}^{n} (a_{ij}\partial_i w \partial_j w + \sum_i b_i \partial_i w w + cw^2)dx, \quad w \in H_0^1(B_1)$$

under the constraint: $\int_{B_1} |w|^2 = 1$ can be reached by a smooth function.