PHD Qualification Exam: Partial Differential Equations 2005

1. Let Ω be a bounded smooth set in $\mathbb{R}^n$ $(n \geq 2)$ and let $L : \mathbb{R}^n \times \mathbb{R} \times \bar{\Omega} \to \mathbb{R}$ be a smooth mapping. Assume in addition that L is bounded below and the mapping $p \to L(p, z, x)$ is convex for each $z \in \mathbb{R}$, $x \in \bar{\Omega} \in \mathbb{R}^n$. Prove that $I(w) := \int_\Omega L(Dw, w, x)dx$ is weakly lower semicontinuous on $W^{1,q}(\Omega)$ $(1 < q < \infty)$.

2. Let $\{a_{ij}\}_{n\times n}$ $(n \geq 2)$ be a positive definite constant matrix. Let B_1 be the unit ball in $\mathbb{R}^n$. Find the Euler-Lagrange equation for

$$I(w) = \int_{B_1} \sum_{i,j=1}^{n} (a_{ij}\partial_i w \partial_j w)dx, \quad w \in H_0^1(B_1)$$

under the constraint: $\int_{B_1} |w|^2 = 1$.

3. Prove that if $\lambda > 0$ is large enough, there exists a function $u \in H^2(B_1) \cap H_0^1(B_1)$ solving

$$\begin{cases} -\Delta u + \lambda u = \sqrt{1 + |Du|^2}, & \text{in} \quad B_1, \\ u = 0, & \text{on} \quad \partial B_1. \end{cases}$$

4. Let Ω be a bounded smooth subset of $\mathbb{R}^n$ $(n \geq 2)$ and let u be a classical solution of

$$\begin{cases} \Delta u(x) + a(x)u(x) = 0 \quad \Omega, \\ u \geq 0, \quad \partial\Omega, \end{cases}$$

where $a(x)$ is a smooth function. Suppose there exists $\phi(x) > 0$ and smooth over $\bar{\Omega}$ such that

$$\Delta\phi + a(x)\phi \leq 0, \quad \Omega.$$

Show that either $u > 0$ or $u \equiv 0$ in Ω.

5. Let $u \in C^2(B_1)$ be a solution of

$$\begin{cases} -\Delta u = e^u, \quad B_1 \subset \mathbb{R}^2, \\ u = 1, \quad \text{on} \quad \partial B_1. \end{cases}$$

Prove that (1): $u > 1$ in B_1, (2): u is radially symmetric and $\frac{du}{dr} < 0$ for all $r \in (0, 1)$.

6. (a) Show there exists a unique minimizer $u \in A$ of

$$I[w] = \int_\Omega (\frac{1}{2}|Dw|^2 - fw)dx$$

where Ω is a bounded smooth subset of $\mathbb{R}^n$ $(n \geq 2)$, $f \in L^2(\Omega)$,

$$A = \{w \in H_0^1(\Omega); \quad |Dw| \leq 1 \quad a.e.\}.$$

(b) Prove

$$\int_\Omega Du \cdot D(w-u)dx \geq \int_\Omega f(w-u)dx$$

for all $w \in A$.

7. Let $u \in W^{1,p}(B_1)$. Prove that $u^+, u^-, |u| \in W^{1,p}(B_1)$ and for almost all points over each region ($u > 0$, $u < 0$, $u = 0$) (Recall $u = u^+ - u^-$).

$$Du^+ = \begin{cases} Du, u > 0, \\ 0, u \leq 0. \end{cases} \quad Du^- = \begin{cases} -Du, u < 0, \\ 0, u \geq 0. \end{cases} \quad D|u| = \begin{cases} Du, u > 0, \\ -Du, u < 0, \\ 0, u = 0. \end{cases}$$

Hint: Let $f_\epsilon(u) = \begin{cases} (u^2+\epsilon^2)^{1/2} - \epsilon, u > 0, \\ 0, u \leq 0. \end{cases}$ For all $\phi \in C_0^1(B_1)$,

$$\int_{B_1} f_\epsilon(u) D\phi dx = -\int_{u>0} \frac{uDu}{\sqrt{u^2+\epsilon^2}} \phi dx.$$

8. Let $a_{ij}(x)$, $b_i(x)$ $(i, j = 1, ..n)$ $c(x)$ be C^∞ functions over B_1 $(n \geq 2)$. Let λ_0, λ_1 be two positive constants satisfying

$$0 < \lambda_0|\xi|^2 \leq a_{ij}(x)\xi_i\xi_j \leq \lambda_1|\xi|^2 \quad \forall x \in B_1 \quad \forall \xi \in \mathbb{R}^n.$$

Suppose $u \in W^{1,q}$ is a weak solution of

$$\sum_{i,j} \partial_i(a_{ij}(x)\partial_j u) + \sum_i b_i(x)\partial_i u(x) + c(x)u = u^2, \quad B_1,$$

Look for the smallest $q_0 > 1$ such that for $q > q_0$, $u \in C^\infty(B_1)$. Prove your statement.

Hint: The following estimates should be used: Suppose $u \in W^{1,q}(B_1)$ verifies

$$\sum_{i,j} \partial_i(a_{ij}(x)\partial_j u) + \sum_i b_i(x)\partial_i u(x) + c(x)u = f(x) \quad B_1$$

in the sense of distribution, $f \in L^q, q \in (1,\infty)$. Then

$$\|u\|_{W^{2,q}(B_{1/2})} \leq C(\|u\|_{L^q(B_1)} + \|f\|_{L^q(B_1)}).$$

If we further know $u \in C^\alpha(B_1)$ and $f \in C^\alpha(B_1)$ for some $\alpha \in (0,1)$, then the following Schauder estimate holds:

$$\|u\|_{C^{2,\alpha}(B_{1/2})} \leq C(\|u\|_{C^\alpha(B_1)} + \|f\|_{C^\alpha(B_1)}).$$