University of Florida (2001)

Ph. D. Examination in Partial Differential Equations

Instructions. Do all problems 1–4. Choose one problem of 5 and 5′ and one problem of 6 and 6′.

(1) Let B be the unit ball in R^3. For which values of α does the function $|x|^\alpha$ belong to $H^1(B)$? Justify your answer.

(2) Let $U \subset R^n$ be open and bounded. Let u_m and v_m be bounded sequences in $H^1(U)$. Show that there exist subsequences u_{m_j} of u_m and v_{m_j} of v_m, and functions $u, v \in H^1(U)$, such that

$$\int_U u_{m_j} v_{m_j} \to \int_U uv,$$

and

$$\int_U D_{x_i}(u_{m_j} v_{m_j}) \to D_{x_i}(uv), \quad i = 1, \ldots, n,$$

as $j \to \infty$.

(3) Let $U \subset R^n$ be open, bounded, and connected with smooth boundary ∂U. Let $u(x,t)$ be a smooth solution to the IBV problem:

$$u_t = \Delta u, \quad (x,t) \in U_T,$$

$$\frac{\partial u}{\partial n} = 0, \quad (x,t) \in \partial U \times [0,T],$$

$$u(x,0) = f(x), \quad x \in U.$$

Let

$$(u)_U(t) = \frac{1}{|U|} \int_U u(x,t)\, dx.$$

Show that

(a) $\frac{d}{dt}(u)_U(t) \equiv 0$.
(b) $u \to (u)_U$ in $L^2(U)$, as $t \to \infty$.

Hint: Consider the equation for $v = u - (u)_U$, and use energy estimate for the equation of v, and the Poincare's inequality.

(4) Let $U \subset R^n$ be open, bounded, and connected with smooth boundary ∂U. Let L be a uniformly elliptic differential operator of the form:

$$Lu = -\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j}.$$

Suppose that f is a bounded function and $u, v \in C^2(\overline{U})$ satisfy

$$Lu = f, \quad Lv \geq 1, \quad x \in U,$$

$$u(x) = 0, \quad v(x) \geq 0, \quad x \in \partial U.$$

Suppose that there exists a point $x_0 \in \partial U$, such that $v(x_0) = 0$.
Prove that there exists a constant $C > 0$, such that

$$|Du(x_0)| \leq C |\frac{\partial v}{\partial n}(x_0)|.$$

(5) Let $U \subset R^n$ be open and bounded with smooth boundary ∂U. We say that a function $u \in H_0^2(U)$ is a weak solution of the biharmonic equation:

$$\Delta^2 u = f, \quad x \in U, \tag{5.1}$$

$$u = 0, \ \frac{\partial u}{\partial n} = 0, \quad x \in \partial U, \tag{5.2}$$

if

$$\int_U \Delta u \Delta v \, dx = \int_U f v \, dx, \text{ for any } v \in H_0^2(U). \tag{5.3}$$

Prove the following statements:

(a) For a given $f \in L^2(U)$ there is an unique solution to the problem (5.3).
(b) If $u \in C^4(\bar{U})$ and $f \in C^0(\bar{U})$ satisfy (5.3), then u satisfies (5.1) at each point $x \in U$.

(5′) Let $U \subset R^n$ be open and bounded. Show that $u \in H_0^1(U)$ is a weak solution to the boundary value problem

$$-\Delta u + u = 1, \quad x \in U,$$

$$u = 0, \quad x \in \partial U,$$

if only if u minimizes

$$E(v) = \int_U \left(|\nabla v|^2 + v^2 - 2v\right) \, dx$$

over $v \in H_0^1(U)$.

(6) Let $U \subset R^n$ be open and bounded.
We say $v \in C^2(\overline{U})$ is *subharmonic* if $-\Delta v \leq 0$ for all $x \in U$.

(a) Prove that for any subharmonic v,

$$v(x) \leq \int_{B(x,r)} v(y) \, dy, \quad \text{for all } B(x,r) \subset U.$$

(b) Let $\phi : R \to R$ be smooth and convex. Suppose that u is harmonic and $v = \phi(u)$. Prove that v is subharmonic.

(6′) Let $U \subset R^n$ be open and bounded with smooth boundary ∂U.

(a) Write the fundamental solution $\Phi(x)$ of Laplace's equation.
(b) Prove that for any point $x \in U$ and any function $u \in C^2(\bar{U})$,

$$u(x) = \int_U \Phi(y - x) \Delta u(y) \, dy$$

$$+ \int_{\partial U} \Phi(y - x) \frac{\partial u}{\partial n}(y) - u(y) \frac{\partial \Phi}{\partial n}(y - x) \, dS(y),$$

where Φ is the fundamental solution of Laplace's equation, and n is the outward unit normal to ∂U.