Numerical Analysis Qualifying Exam (May 11, 2012). *Answer any 8 questions.*

1. Suppose that A and $B \in \mathbb{C}^{n\times n}$ are Hermitian matrices. If $\sigma(A)$ denotes the largest eigenvalue of A, then show that
$$\sigma(A+B) \le \sigma(A) + \sigma(B).$$

2. Suppose that $A \in \mathbb{C}^{n\times n}$ is an invertible matrix, $u \in \mathbb{C}^n$, and $v \in \mathbb{C}^n$. Let v^* denote the conjugate transpose of v.

   (a) Show that
$$\det(A - uv^*) = (1 - v^*A^{-1}u)\det A.$$

   (b) Show that $A - uv^*$ is invertible if and only if $v^*A^{-1}u \ne 1$. Moreover, if $v^*A^{-1}u \ne 1$, then
$$(A - uv^*)^{-1} = A^{-1} + \left(\frac{1}{1 - v^*A^{-1}u}\right) A^{-1}uv^*A^{-1}.$$

3. (a) Suppose p and $q \in \mathbb{R}$ with p and q positive and $p^{-1} + q^{-1} = 1$. Show that for any matrix $A \in \mathbb{C}^{m\times n}$, we have $\|A\|_p = \|A^*\|_q$.

   (b) Prove that
$$\|A\|_2^2 \le \|A\|_p \|A\|_q$$
   for any $A \in \mathbb{C}^{m\times n}$ and any positive p and $q \in \mathbb{R}$ with $p^{-1} + q^{-1} = 1$.

4. (a) For any matrices A and B, show that the nonzero eigenvalues of AB and BA are the same.

   (b) If AB is normal, $\|\cdot\|_2$ is the 2-norm of a matrix, and $\|\cdot\|$ is an induced matrix norm, then show that $\|AB\|_2 \le \|BA\|$.

5. Let P and Q be two $m \times m$ orthogonal projectors. Prove that $\|P - Q\|_2 \le 1$.

6. Consider the following minimization problem:
$$\tau_n = \inf_{\deg Q < n} \left(\max_{x\in[-1,1]} |x^n + Q(x)| \right).$$
   Prove that $\tau_n = \frac{1}{2^{n-1}}$, the infimum is in fact a minimum, attained at a unique Q^* satisfying
$$x^n + Q^*(x) = \frac{1}{2^{n-1}} T_n(x),$$
   where $\{T_n\}$ are the Chebyshev's polynomials.

7. Design an algorithm with a cubic rate of convergence for computing the quantity $\sqrt{5}$. Prove that your algorithm is indeed cubic. Find an interval $[a, b] \subset [0, +\infty)$ such that any iterative sequence starting in $[a, b]$ will converge to $\sqrt{5}$.

8. State the classical Hermite interpolation problem. Prove that the problem is well-posed, i.e. prove the existence and uniqueness of solutions. Derive the error formula for the interpolating polynomial.

9. Describe the Simpson's Rule for numerical integration. State (without proof) the error formula of the Trapezoidal Rule. Show that the error bound cannot be improved.

10. Let $w(x) > 0$ be integrable on $[a, b]$. Define an orthogonal polynomial family $\{\phi_n\}$ on $[a, b]$ with weight $w(x)$. Prove that between two consecutive zeros of ϕ_n there is exactly one zero of ϕ_{n-1}. You may use the triple recursion formula without proof.