Numerical Analysis Qualifying Exam (September, 2011). *Answer any 8 questions.*

1. (a) For any matrices A and B, show that the nonzero eigenvalues of AB and BA are the same. (b) If AB is normal, $\|\cdot\|_2$ is the 2-norm of a matrix, and $\|\cdot\|$ is an induced matrix norm, then show that $\|AB\|_2 \leq \|BA\|$.

2. (a) Suppose p and $q \in \mathbb{R}$ with p and q positive and $p^{-1} + q^{-1} = 1$. Show that for any matrix $A \in \mathbb{C}^{m\times n}$, we have $\|A\|_p = \|A^*\|_q$. (b) Prove that

$$\|A\|_2^2 \leq \|A\|_p \|A\|_q$$

for any $A \in \mathbb{C}^{m\times n}$ and any positive p and $q \in \mathbb{R}$ with $p^{-1} + q^{-1} = 1$.

3. Let P and Q be two $m \times m$ orthogonal projectors. Prove that $\|P - Q\|_2 \leq 1$.

4. Let P and Q be Hermitian positive definite matrices. Prove that

$$x^* P x \leq x^* Q x \text{ for all } x \in \mathbb{C}^n$$

if and only if

$$x^* Q^{-1} x \leq x^* P^{-1} x \text{ for all } x \in \mathbb{C}^n.$$

5. Suppose A is a Hermitian positive definite matrix split into $A = C + C^* + D$ where D is also Hermitian positive definite. Prove that $B = C + \omega^{-1} D$ is invertible whenever $0 < \omega < 2$. Consider the iteration $x_{n+1} = x_n + B^{-1}(b - Ax_n)$, with any initial iterate x_0. Prove that x_n converges to $x = A^{-1}b$ whenever $0 < \omega < 2$.

6. Let $f : R \to R$ be a contraction with constant $\lambda \in (0, 1)$. Prove that there exists a unique fixed point α of f. Prove the inequality

$$|f^n(x) - \alpha| \leq \frac{\lambda^n}{1-\lambda} |f(x) - x|, \quad \forall x \in R, \ \forall n \in N.$$

7. Derive the two point Gaussian quadrature for approximating $\int_{-1}^{1} f(x)x^2 \, dx$.

8. State the Hermite interpolation problem. Prove that the problem is well-posed, i.e. prove the existence and uniqueness of solutions. Derive the error formula for the interpolating polynomial.

9. Describe the Simpson's rule for numerical integration ($n = 3$). State and prove the error formula of the Simpson's rule.

10. Let $w(x) > 0$ be integrable on $[a, b]$. Define an orthogonal polynomial family $\{\phi_n\}$ on $[a, b]$ with weight $w(x)$. Prove that (i) $\phi_n(x)$ has n distinct roots in the interval (a, b), and (ii) between any two consecutive zeros of ϕ_n there is a zero of ϕ_{n-1}.