Numerical Analysis Qualifying Exam (January, 2011). *Answer any 8 questions.*

1. Suppose A and B are square matrices such that AB is normal. Prove that $\|AB\|_2 \leq \|BA\|$. (We use $\|\cdot\|_2$ to denote the spectral norm and $\|\cdot\|$ to denote any induced matrix norm.)

2. If P is a projector that is neither 0 nor the identity, prove that $\|P\| = \|I - P\|$ in any norm induced by a vector norm generated by an inner product.

3. Suppose A is a Hermitian positive definite matrix split into $A = C + C^* + D$ where D is also Hermitian positive definite. Prove that $B = C + \omega^{-1}D$ is invertible whenever $0 < \omega < 2$. Consider the iteration $x_{n+1} = x_n + B^{-1}(b - Ax_n)$, with any initial iterate x_0. Prove that x_n converges to $x = A^{-1}b$ whenever $0 < \omega < 2$.

4. If a square matrix A can be block partitioned as $A = \left[\begin{smallmatrix} 0 & M \\ N & 0 \end{smallmatrix}\right]$, prove that $\sigma(A) = \{z \in \mathbb{C} : z^2 \in \sigma(MN) \cup \sigma(NM)\}$.

5. Let $A \in \mathbb{R}^{N\times N}$ and $b \in \mathbb{R}^N$. Consider the following iteration for solving $Ax = b$, that computes x_{n+1}, given $x_n \in \mathbb{R}^N$, as follows (in m intermediate steps): Setting $x_{n+1}^{(0)} = x_n$, for $\ell = 1, 2, \ldots, m$, compute $x_{n+1}^{(\ell)} = x_{n+1}^{(\ell-1)} + \tau_\ell(b - Ax_{n+1}^{(\ell-1)})$. Then, define $x_{n+1} = x_{n+1}^{(m)}$. There is a linear operator E such that $x_{n+1} - x = E(x_n - x)$. Give a formula for E. Suppose A is Hermitian and positive definite with spectral condition number κ. Prove that there are real values of the m parameters τ_ℓ such that

$$\rho(E) \leq 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m.$$

6. (The Aitken accelerated convergence algorithm). Let $\alpha = \phi(\alpha)$ be a fixed point of $\phi(x)$. Suppose that ϕ is twice continuously differentiable with $\phi'(\alpha) \neq 1$. Consider the iteration

$$\begin{aligned} x_{n+1} &= \Phi(x_n) \\ \Phi(x) &= \phi(\phi(x)) + \frac{H(x)}{1 - H(x)}\Big(\phi(\phi(x)) - \phi(x)\Big) \\ H(x) &= \frac{\phi(\phi(x)) - \phi(x)}{\phi(x) - x} \end{aligned}$$

Show that the iteration $x_{n+1} = \Phi(x_n)$ is quadratically convergent to α when the starting guess x_0 is sufficiently close to α.

7. Let n be a positive integer, $h = 1/n$, and consider the grid of points (ih, jh) for $i, j = 0, 1, \ldots, n$. Let A be the finite difference operator with the "5-point stencil" discretizing the Laplace operator $-\Delta$, with zero Dirichlet boundary conditions on this grid. Describe it. Prove that the spectrum of A consists of the numbers

$$\lambda_{lm} = 4h^{-2}(\sin^2(l\pi h/2) + \sin^2(m\pi h/2)),$$

for all $l, m = 1, \ldots, n-1$.

8. Let S_n^1 denote the space of linear splines based on knots $a = t_0 < t_1 < \cdots < t_n = b$, i.e., $S_n^1 = \{v : v|_{[t_j, t_{j+1}]}$ is linear for all $j = 0, 1, \ldots, n-1$ and v is continuous on $[a, b]\}$. Let $s_f \in S_n^1$ interpolate a continuous function f at the knots. Prove that $\|f - s_f\|_\infty \leq 2\|f - s\|_\infty$ for any $s \in S_n^1$.

9. Given that the Newton iteration for finding a root r of f converges, $f \in C^3(\mathbb{R})$, and $f(r) = f'(r) = 0 \neq f''(r)$, prove that the convergence cannot be quadratic. Suggest a modification that restores quadratic convergence for smooth f.

10. Let $x_0, x_1, \ldots, x_n$ be distinct numbers and for each i, let ℓ_i denote the product of all $(x_i - x_j)^{-1}$ for all $j \neq i$, $j = 0, \ldots, n$. Express the number $q_f = f(x_0)\ell_0 + f(x_1)\ell_1 + \cdots + f(x_n)\ell_n$ as a divided difference of f. Then show that $q_f = 0$ whenever f is a polynomial of degree $n - 1$.