Numerical Analysis Qualifying Exam (May 2010). *Answer any 8 questions.*

1. If P is a projector that is neither 0 nor the identity, prove that $\|P\| = \|I - P\|$ in any norm induced by a vector norm generated by an inner product.

2. Let $M = \left[\begin{smallmatrix} A & B^t \\ B & C \end{smallmatrix}\right]$ be a real symmetric and positive definite matrix with an invertible A. Prove that the Schur complement $S = C - BA^{-1}B^t$ is also symmetric and positive definite.

3. Suppose A is a Hermitian positive definite matrix split into $A = C + C^* + D$ where D is also Hermitian positive definite. Prove that $B = C + \omega^{-1}D$ is invertible whenever $0 < \omega < 2$. Consider the iteration $x_{n+1} = x_n + B^{-1}(b - Ax_n)$, with any initial iterate x_0. Prove that x_n converges to $x = A^{-1}b$ whenever $0 < \omega < 2$.

4. If a square matrix A can be block partitioned as $A = \left[\begin{smallmatrix} 0 & M \\ N & 0 \end{smallmatrix}\right]$, prove that $\sigma(A) = \{z \in \mathbb{C} : z^2 \in \sigma(MN) \cup \sigma(NM)\}$.

5. Suppose A is a Hermitian positive definite matrix with eigenvalues

$$0 < \lambda_0 \ll \lambda_1 \le \lambda_2 \le \cdots \le \lambda_N.$$

Let $\kappa = \lambda_N/\lambda_0$ and $\kappa' = \lambda_N/\lambda_1$. Let e_n denote the error in the nth step of the conjugate gradient algorithm applied to $Ax = b$, and $\|y\|_A = \langle Ay, y\rangle^{1/2}$. Prove that

$$\|e_n\|_A \le 2\kappa \left(\frac{\sqrt{\kappa'} - 1}{\sqrt{\kappa'} + 1} \right)^{n-1} \|e_0\|_A.$$

6. Let n be a positive integer, $h = 1/n$, and consider the grid of points (ih, jh) for $i, j = 0, 1, \ldots, n$. Let A be the finite difference operator with the "5-point stencil" discretizing the Laplace operator $-\Delta$, with zero Dirichlet boundary conditions on this grid. Describe it. What are the eigenvalues of A? Prove that the spectral condition number of A is $\kappa(A) = \cot^2(\pi h/2)$.

7. Let $x_0, x_1, \cdots, x_n$ be distinct points in a finite interval $[a, b]$ and $f \in C^1[a, b]$. Show that for any given $\varepsilon > 0$ there exists a polynomial p such that $\|f - p\|_\infty < \varepsilon$ and $p(x_i) = f(x_i)$ for all $i = 0, 1, \cdots, n$ (where $\|\cdot\|_\infty$ denotes the $L^\infty(a, b)$-norm).

8. Let $w(x)$ be a positive integrable function on $[a, b]$. Consider the quadrature

$$\int_a^b f(x)\, w(x)\, dx \;\approx\; \sum_{k=0}^{n} \big(A_k f(x_k) + B_k f'(x_k) + C_k f''(x_k) \big)$$

for any f with continuous first and second derivatives (f' and f'', respectively). Give conditions on A_k, B_k, C_k and x_k so that the quadrature has precision $4n + 3$.

9. Given that the Newton iteration for finding a root r of f converges, $f \in C^2(\mathbb{R})$, and $f(r) = f'(r) = 0 \ne f''(r)$, prove that the convergence cannot be quadratic. Suggest a modification that restores quadratic convergence for smooth f.

10. Let $x_0, x_1, \ldots, x_n$ be distinct numbers and for each i, let ℓ_i denote the product of all $(x_i - x_j)^{-1}$ for all $j \ne i$, $j = 0, \ldots, n$. Express the number $q_f = f(x_0)\ell_0 + f(x_1)\ell_1 + \cdots + f(x_n)\ell_n$ as a divided difference of f. Then show that $q_f = 0$ whenever f is a polynomial of degree $n - 1$.