Numerical Analysis Qualifying Exam (Spring 2010). *Answer any 8 questions.*

1. Prove that a projector is normal if and only if it is self-adjoint.

2. Let $A \in \mathbb{R}^{N\times N}$ and $b \in \mathbb{R}^N$. Consider the following iteration for solving $Ax = b$, that computes x_{n+1}, given $x_n \in \mathbb{R}^N$, as follows (in m intermediate steps): Setting $x_{n+1}^{(0)} = x_n$, for $\ell = 1, 2, \ldots, m$, compute $x_{n+1}^{(\ell)} = x_{n+1}^{(\ell-1)} + \tau_\ell(b - Ax_{n+1}^{(\ell-1)})$. Then, define $x_{n+1} = x_{n+1}^{(m)}$. There is a linear operator E such that $x_{n+1} - x = E(x_n - x)$. Give a formula for E. Suppose A is Hermitian and positive definite with spectral condition number κ. Prove that there are real values of the m parameters τ_ℓ such that
$$\rho(E) \le 2\left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^m.$$

3. Suppose A and B are square matrices such that AB is normal. Prove that $\|AB\|_2 \le \|BA\|$. (We use $\|\cdot\|_2$ to denote the spectral norm and $\|\cdot\|$ to denote any induced matrix norm.)

4. Let $A \in \mathbb{C}^{m\times m}$, and a_j be its j-th column. Prove that $|\det A| \le \prod_{j=1}^{m} \|a_j\|_2$.

5. Suppose A is a Hermitian positive definite matrix split into $A = C + C^* + D$ where D is also Hermitian positive definite. Show that $B = C + \omega^{-1}D$ is invertible. Consider the iteration $x_{n+1} = x_n + B^{-1}(b - Ax_n)$, with any initial iterate x_0. Prove that x_n converges to $x = A^{-1}b$ whenever $0 < \omega < 2$.

6. Let $x_0, x_1, \cdots, x_n$ be distinct points in a finite interval $[a, b]$ and $f \in C^1[a, b]$. Show that for any given $\varepsilon > 0$ there exists a polynomial p such that $\|f - p\|_\infty < \varepsilon$ and $p(x_i) = f(x_i)$ for all $i = 0, 1, \cdots, n$ (where $\|\cdot\|_\infty$ denotes the $L^\infty(a, b)$-norm).

7. Let $p > 0$ and $x = \sqrt{p + \sqrt{p + \sqrt{p + \cdots}}}$, where all the square roots are positive. Design a fixed point iteration $x_{n+1} = F(x_n)$ that converges to x. You should write down a specific F. Obtain a sufficient condition on the initial iterates for (global) convergence of the iteration.

8. Let n be a positive integer, $h = 1/n$, and consider the grid of points (ih, jh) for $i, j = 0, 1, \ldots, n$. Let A be the finite difference operator with the "5-point stencil" discretizing the Laplace operator $-\Delta$, with zero Dirichlet boundary conditions on this grid. Describe it. Prove that the spectrum of A consists of the numbers
$$\lambda_{lm} = 4h^{-2}(\sin^2(l\pi h/2) + \sin^2(m\pi h/2)),$$
for all $l, m = 1, \ldots, n-1$.

9. Let x_m and x_{m+1} be two successive (complex) iterates when Newton's method is applied to a polynomial $p(z)$ of degree n. Prove that there is a zero of $p(z)$ in the disk $\{z \in \mathbb{C} : |z - x_m| \le n|x_{m+1} - x_m|\}$.

10. Let $w(x)$ be a positive integrable function on $[a, b]$. Consider the quadrature
$$\int_a^b f(x)\, w(x)\, dx \;\approx\; \sum_{k=0}^{n} \big(A_k f(x_k) + B_k f'(x_k) + C_k f''(x_k)\big)$$
for any f with continuous first and second derivatives (f' and f'', respectively). Give conditions on A_k, B_k, C_k and x_k so that the quadrature has precision $4n + 3$.