Numerical Analysis Qualifying Exam (September, 2009). *Answer any 8 questions.*

1a. Suppose u and $v \in \mathbb{C}^n$. Derive a formula for the p-norm of uv^*, $p \geq 1$.

1b. Let $A = U\Sigma V^*$ be the singular value decomposition of A, where the diagonal elements of Σ are in decreasing order:

$$\sigma_1 \geq \sigma_2 \geq \sigma_3 \geq \dots$$

Show that $\|A\|_2 = \sigma_1$ and

$$\|A\|_F = \sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3 + \dots}$$

2. If $A = QR$ is the QR factorization of A, show that r_{ii} is the distance from the i-th column of A to the space spanned by the first $i-1$ columns of A.

3. Let $\|\cdot\|_p$ for $1 \leq p \leq \infty$ denote the norm on $m \times n$ matrices induced by the ℓ^p-norm. For any positive p and q with $p^{-1} + q^{-1} = 1$, show that

$$\|A\|_2^2 \leq \|A\|_p \|A\|_q, \quad \text{for all } A \in \mathbb{C}^{m\times n}.$$

Hint: Recall that for a Hermitian matrix H, $\|H\|_2$ is the absolute largest eigenvalue. As a result $\|H\|_2 \leq \|H\|$ for any matrix norm induced by a vector norm.

4. Prove Gerschgorin's theorem: If $A \in \mathbb{C}^{n\times n}$, then each eigenvalue of A lies in the union of the disks

$$D_i = \{\lambda \in \mathbb{C} : |\lambda - a_{ii}| \leq \sum_{j\neq i} |a_{ij}|\}.$$

Moreover, if m disks form a connected domain that is disjoint from the other $n-m$ disks, then there are precisely m eigenvalues of A within this domain.

5. Let P and Q be two $m \times m$ orthogonal projectors. Prove that $\|P - Q\|_2 \leq 1$.

6. Assume that $g(x)$ is continuously differentiable on $[a,b]$, that $g([a,b]) \in [a,b]$, and that

$$\lambda = Max_{a\leq x\leq b}|g'(x)| < 1.$$

Prove that the following are true:
(i) $x = g(x)$ has a unique solution α in $[a,b]$.
(ii) For any initial choice x_0 in $[a,b]$, the iteration $x_{n+1} = g(x_n)$ has the property that $lim_{n\to\infty} x_n = \alpha$.
(iii)

$$lim_{n\to\infty} \frac{\alpha - x_{n+1}}{\alpha - x_n} = g'(\alpha).$$

7. Assume that $f(x)$, $f'(x)$, and $f''(x)$ are continuous in $[a,b]$, and that for some $\alpha \in (a,b)$, we have $f(\alpha) = 0$ and $f'(\alpha) \neq 0$. Show that if x_0 is chosen close enough to α, the iterates

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

converge to α. Moreover

$$\lim_{n\to\infty} \frac{\alpha - x_{n+1}}{(\alpha - x_{n+1})^2} = -\frac{f''(\alpha)}{2f'(\alpha)}.$$

8. Let $x_0, x_1, x_2....x_n$ be distinct real numbers and let f be a given real-valued function with $n+1$ continuous derivatives. Let I be an interval containing all of the x_k, and t. Show that there is a $\xi \in I$ such that

$$f(t) - \sum_{k=0}^{n} f(x_k) l_k(t) = \frac{(t-x_0)(t-x_1)...(t-x_n)}{(n+1)!} f^{(n+1)}(\xi)$$

where l_k is the Lagrange interpolating polynomial which is 1 at x_k and 0 at the other x_j, $j \neq k$.

9. Let $I_n(f) = \sum_{j=0}^{n} w_{j,n} f(x_{j,n})$ be a sequence of approximations to $I(f) = \int_a^b f(x)dx$. Let F be a family of functions which is dense in $C[a,b]$, and suppose that (a) $I_n(f) \to I(f)$ for all $f \in F$, and (b) $B = sup_n \sum_{j=0}^{n} |w_{j,n}| < \infty$. Show that $I_n(f) \to I(f)$ for all $f \in C[a,b]$.

10. Consider the differential equation $y' = f(t,y), y(0) = y_0$, where $|f(t,y_1) - f(t,y_2)| \leq K|y_1 - y_2|$ for all y_1 and $y_2 \in \mathbb{R}$. Show that a multistep method

$$y_{n+1} = \sum_{j=0}^{p} a_j y_{n-j} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j})$$

has global error $\tau(h) = O(h^m)$ for all $y(x)$ which are (m+1) times continuously differentiable if and only if

$$\sum_{j=0}^{p} a_j = 1,$$

and

$$\sum_{j=0}^{p} (-j)^i a_j + i \sum_{j=-1}^{p} (-j)^{i-1} b_j = 1$$

for $i = 1, 2, 3, ..., m$.