**Numerical Analysis Qualifying Exam** (January, 2009). *Answer any 8 questions.*

1. Let $M = \left[\begin{smallmatrix} A & B^t \\ B & C \end{smallmatrix}\right]$ be real symmetric and positive definite matrix. Let $S = C - BA^{-1}B^t$ be the Schur complement. Prove that $S$ is symmetric and positive definite.

2. Let $\|\cdot\|_p$ for $1 \le p \le \infty$ denote the norm on $m \times n$ matrices induced by the $\ell^p$-norm on vectors in $\mathbb{C}^m$ and $\mathbb{C}^n$. Prove that
$$\|A\|_2 \le \|A\|_1 \, \|A\|_\infty, \qquad \text{for all } A \in \mathbb{C}^{m\times n}.$$

3. Show that if $P$ and $Q$ be Hermitian positive definite matrices satisfying $x^*Px \le x^*Qx$, for all $x \in \mathbb{C}^n$, then $\|P\|_F \le \|Q\|_F$.

4. Let $T$ be any square matrix and let $\|\cdot\|$ denote any induced norm. Prove that $\lim_{n\to\infty} \|T^n\|^{1/n}$ exists and equals $\inf_{n=1,2,\dots} \|T^n\|^{1/n}$.

5. Let $P$ and $Q$ be two $m \times m$ orthogonal projectors. Prove that $\|P - Q\|_2 \le 1$.

6. Let $x_0, x_1, \cdots, x_n$ be distinct points in a finite interval $[a, b]$ and $f \in C^1[a, b]$. Show that for any given $\varepsilon > 0$ there exists a polynomial $p$ such that $\|f - p\|_\infty < \varepsilon$ and $p(x_i) = f(x_i)$ for all $i = 0, 1, \cdots, n$ (where $\|\cdot\|_\infty$ denotes the $L^\infty(a, b)$-norm).

7. Let $\hat{f}(s)$ be the continuous Fourier transform of $f(t) \in L^2[-\infty, \infty]$. Suppose further that $\hat{f}(s) = 0$ for $|s| > \pi$. Derive the interpolation formula
$$f(t) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

8. Let $L_n(f)$ be the Lagrange polynomial interpolating a function $f$ at nodes $a = x_0 < x_1 \dots .. < x_n = b$. (a) Give the formula for $L_n(f)$. (b) Prove that there is a unique polynomial of degree at most $n$ interpolating $f$ at the nodes. (c) State and prove the formula for the error $f(x) - L_n(f)(x)$ when $f(x) \in C^{n+1}[a, b]$,

9. Assume that $g$ is continuously differentiable real-valued function and $a \le g(x) \le b$ on $[a, b]$. Show the following:
   (a) There is an $\alpha \in [a, b]$ such that $g(\alpha) = \alpha$
   (b) If $|g'(x)| < 1$ on $[a, b]$, then there is only one fixed point on the interval $[a, b]$.
   (c) If we generate iterates using the recurrence $x_{n+1} = g(x_n)$ starting from some $x_0 \in [a, b]$, then we have
   $$|\alpha - x_n| \le \frac{\lambda^n}{1-\lambda} |x_1 - x_0| \quad \text{where} \quad \lambda = \max_{x\in[a,b]} |g'(x)|.$$

10. Let $\{\phi_n(x) : n \ge 0\}$ be an orthogogonal family of polynomials on the interval $(a, b)$ with weight function $w(x) \ge 0$ for all $x \in [a, b]$. Show that the polynomial $\phi_n(x)$ has exactly $n$ distinct real roots in the open interval $(a, b)$.