Numerical Analysis Qualifying Examination

May 2006

Do any eight of the ten problems below.

1. Let $x_0, \ldots, x_n$ be distinct real points. Let

$$\{l_j(x)\}_{j=0,\ldots,n}$$

be the Lagrange basis functions for these points. Prove that

$$\sum_{j=0}^{n} l_j(x) = 1.$$

for all x.

2. Suppose you want to solve numerically the ode

$$y' = f(x, y) \quad y(0) = y_0.$$

(a) By applying the midpoint rule to the integral in the equation

$$y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(s, y(s))ds,$$

derive the midpoint method for odes.

(b) Find the order of the local truncation error for this method.

(c) Determine if this method is strongly stable, weakly stable or unstable.

3. Given x_0, x_1 and smooth $f(x)$, give a divided difference type formula for the cubic polynomial which satisfies:

$$p(x_0) = f(x_0)$$
$$p'(x_0) = f'(x_0)$$
$$p''(x_0) = f''(x_0)$$
$$p(x_1) = f(x_1)$$

Also give a divided difference type error formula for $f(x) - p(x)$.

4. (a) Find the linear least squares approximation to $f(x) = x^2$ on the interval $[0, 2]$ by optimizing the least squares error over a and b in $r^*(x) = ax + b$.

(b) Find the first two orthonormal polynomials $\phi_0(x), \phi_1(x)$ on $[0, 2]$ (weight function $w(x) = 1$). Use these to find the linear least squares approx to $f(x) = x^2$, and check this result agrees with the previous problem.

5. Consider the root-finding iteration defined by

$$x_{n+1} = x_n - f(x_n) \left[\frac{f(x_n)}{f(x_n + f(x_n)) - f(x_n)} \right].$$

This is called Steffensen's method. Show that the method converges quadratically when applied to $f(x) = x^2 - a$. (You may assume x_0 is close enough to the root.)

6. (a) Evaluate the p-norm of a diagonal matrix, $1 \le p \le \infty$.
   (b) Evaluate the p-norm of a rank one matrix $\mathbf{u}\mathbf{v}^*$, $1 \le p \le \infty$, $\mathbf{u} \in \mathbb{C}^m$ and $\mathbf{v} \in \mathbb{C}^n$.
   (c) Suppose $\mathbf{A} \in \mathbb{C}^{m \times n}$ can be expressed

$$\mathbf{A} = \sum_{k=0}^{r} \sigma_k \mathbf{u}_k \mathbf{v}_k^*,$$

   where the vectors $\{\mathbf{u}_k : 1 \le k \le r\}$ and $\{\mathbf{v}_k : 1 \le k \le r\}$ are orthonormal. What is $\|\mathbf{A}\|_2$?

7. (a) Given $\mathbf{A} \in \mathbb{C}^{m \times n}$, show that $\mathbf{A}^*\mathbf{A}$ is positive semidefinite and $\mathbf{A}^*\mathbf{A}$ is positive definite if and only if the columns of $\mathbf{A}$ are linearly independent.
   (b) If $\mathbf{A}$ is a Hermitian, positive semidefinite matrix, show that the diagonal contains the largest in magnitude element of $\mathbf{A}$. Hint: show that $|a_{ij}| \le \max\{|a_{ii}|, |a_{jj}|\}$.

8. Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ be nonsingular. Show that $\mathbf{A}$ has an LU factorization if and only if for each k with $1 \le k \le n$, the upper-left $k \times k$ block $\mathbf{A}_{1:k,1:k}$ is nonsingular. Prove that this LU factorization is unique.

9. Consider the Arnoldi iteration on the Krylov spaces $\mathcal{K}_k = \text{span}\{\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \ldots \mathbf{A}^{k-1}\mathbf{b}\}$ for some $\mathbf{b} \in \mathbb{C}^m$ and $\mathbf{A} \in \mathbb{C}^{m \times m}$ (given at side). Suppose the algorithm proceeds without breakdown until for some $n < m$, it encounters $h_{n+1,n} = 0$.
   (a) Show that $\mathcal{K}_n$ is an invariant subspace of $\mathbf{A}$, i.e., $\mathbf{A}\mathcal{K}_n \subseteq \mathcal{K}_n$.
   (b) Show that $\mathcal{K}_n = \mathcal{K}_{n+1} = \mathcal{K}_{n+2} = \cdots$.
   (c) Let $\mathbf{H}_n$ be the $n \times n$ Hessenberg matrix whose ij-th entry ($i \le j+1$) is the number h_{ij} computed in the algorithm. Show that each eigenvalue of $\mathbf{H}_n$ is an eigenvalue of $\mathbf{A}$.

ALGORITHM 1 (ARNOLDI ITERATION)
(a) Set $\mathbf{q}_1 = \mathbf{b}/\|\mathbf{b}\|_2$.
(b) For $n = 1, 2, 3, \ldots$ do:
   i. Set $\mathbf{v} = \mathbf{A}\mathbf{q}_n$.
   ii. For $j = 1, 2, \ldots, n$ do:
      A. Set $h_{jn} = \mathbf{q}_j^* v$.
      B. Replace $\mathbf{v}$ by $\mathbf{v} - h_{jn}\mathbf{q}_j$.
   iii. Set $h_{n+1,n} = \|\mathbf{v}\|_2$.
   iv. Set $\mathbf{q}_{n+1} = \mathbf{v}/h_{n+1,n}$.

10. Consider an invertible linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$ and a perturbation $(\mathbf{A} + \delta\mathbf{A})(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b}$. Assuming $\|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\| < 1$, derive the condition estimate

$$\frac{\|\delta\mathbf{x}\|}{\|\mathbf{x}\|} \le \left(\frac{\|\mathbf{A}\|\|\mathbf{A}^{-1}\|}{1 - \|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\|}\right) \frac{\|\delta\mathbf{A}\|}{\|\mathbf{A}\|}.$$

For given $\mathbf{A}$ and $\mathbf{x}$ and for the 2-norm, what choice of $\delta\mathbf{A}$ yields the following equality:

$$\|\mathbf{A}^{-1}\delta\mathbf{A}\mathbf{x}\| = \|\mathbf{A}^{-1}\|\|\delta\mathbf{A}\|\|\mathbf{x}\|$$