Numerical Analysis Qualifying Examination

May 2005

Do any eight of the ten problems below.

1. (a) Consider the matrix $\mathbf{A} = \mathbf{u}\mathbf{v}^*$ where $\mathbf{u}$ and $\mathbf{v} \in \mathbb{C}^n$. Under what condition on $\mathbf{u}$ and $\mathbf{v}$ is $\mathbf{A}$ a projector?

   (b) Show that the Householder matrix $\mathbf{H} = \mathbf{I} - 2\mathbf{w}\mathbf{w}^*$ where $\|\mathbf{w}\| = 1$ is unitary.

   (c) Given a vector $\mathbf{x} \in \mathbb{C}^n$ and an integer k with $1 < k < n$, derive a formula for a Householder matrix with the property that $(\mathbf{Hx})_i = 0$ for $i > k$ and $(\mathbf{Hx})_i = \mathbf{x}_i$ for $i < k$. Be sure to choose the signs so that the formula is numerically stable.

2. (a) Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ with n linearly independent columns, derive the formula (the normal equation) for the least squares solution to the possibly overdetermined linear system $\mathbf{Ax} = \mathbf{b}$.

   (b) Given a matrix $\mathbf{A} \in \mathbb{C}^{m \times n}$ with m linearly independent rows, derive the formula (the analogue of the normal equation) for the minimum norm solution $\mathbf{Ax} = \mathbf{b}$.

   (c) Suppose you are given the singular value decomposition (SVD) $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^*$. Derive a single formula for the least squares/ minimum norm solution to $\mathbf{Ax} = \mathbf{b}$ which is expressed in terms of $\mathbf{U}$, $\mathbf{\Sigma}$, and $\mathbf{V}$.

3. Consider the invertible linear system $\mathbf{Ax} = \mathbf{b}$. Derive a tight bound for the condition number with respect to perturbations in $\mathbf{b}$. If $\mathbf{A}(\mathbf{x} + \delta\mathbf{x}) = \mathbf{b} + \delta\mathbf{b}$ is the perturbed system, then with the 2-norm, for which choice of $\mathbf{b}$ *and* for which perturbation $\delta\mathbf{b}$ is this bound tight?

4. Let

$$\mathbf{C} = \begin{bmatrix} -a_1 & -a_2 & \cdots & -a_{n-1} & -a_n \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}.$$

   (a) What is the characteristic polynomial of $\mathbf{C}$?

   (b) Consider any $\mathbf{A} \in \mathbb{C}^{n \times n}$ and $\mathbf{b} \in \mathbb{C}^n$. Let the characteristic polynomial of $\mathbf{A}$ be $p(\lambda) = \lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \dots a_{n-1}\lambda + a_n$,

$$\mathbf{K} = [\mathbf{b}, \mathbf{Ab}, \mathbf{A}^2\mathbf{b}, \dots, \mathbf{A}^{n-1}\mathbf{b}], \quad \text{and} \quad \mathbf{R} = \begin{bmatrix} 1 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 1 & a_1 & \cdots & a_{n-2} \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & a_1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix}.$$

   Show that if $\mathbf{K}$ is invertible, then

$$(\mathbf{KR})^{-1}\mathbf{A}\,(\mathbf{KR}) = \mathbf{C} \quad \text{and} \quad (\mathbf{KR})^{-1}b = e.$$

   (Suggestion: Prove that $\mathbf{A}\,(\mathbf{KR}) = (\mathbf{KR})\,\mathbf{C}$ using $p(\mathbf{A}) = 0$.)

5. Let $\mathbf{A} = \mathbf{QR}$ be the factorization of a $\mathbf{A}$ into the product of a unitary matrix and a triangular matrix. Suppose that the columns of $\mathbf{A}$ are linearly independent. Show that $|r_{kk}|$ is the distance from the k-th column of $\mathbf{A}$ to the linear space spanned by the first $k-1$ columns of $\mathbf{A}$.

6. Show that $x = \pi + 0.5\sin(x/2)$ has a unique solution α in $\mathbb{R}$. Give a fixed point iteration by which one can compute an approximation to α numerically and prove that it converges at order one.

7. The Lagrange interpolation problem is to find a polynomial $p_n(x)$ of degree n such that

$$p_n(x_i) = y_i, \quad i = 0 \ldots n$$

where $x_0, \ldots, x_n$ are distinct real points and $y_0, \ldots, y_n$ are given data. Prove that this problem always has a unique solution.

8. Consider the quadrature that approximates

$$\int_a^b f(x)dx$$

by

$$I_3(f) := \int_a^b P_3(x)dx,$$

where $P_3(x)$ is the cubic Hermite interpolant to $f(x)$ at the points $x = a, b$. Obtain an error formula for this approximation. (This is called the corrected trapezoid rule.)

9. Consider multistep methods of the form

$$y_{n+1} = y_{n-q} + h \sum_{j=-1}^{p} b_j f(x_{n-j}, y_{n-j})$$

with $q \geq 1$. Show that such methods do not satisfy the strong root condition. Find an example with $q = 1$ that is relatively stable.

10. Let S_n^1 denote the space of linear splines based on knots $a = t_0 < t_1 < \cdots < t_n = b$, i.e.,

$$S_n^1 = \{v : v|_{[t_j, t_{j+1}]} \text{ is linear for all } j = 0, 1, \ldots, n-1 \text{ and } v \text{ is continuous on } [a, b]\}.$$

Let $s_f \in S_n^1$ interpolate a continuous function f at the knots.

(a) Prove that $\|s_f\|_\infty \leq \|f\|_\infty$. (Here $\|v\|_\infty = \max_{x\in[a,b]} |v(x)|$.)

(b) Prove that $\|f - s_f\|_\infty \leq 2\|f - s\|_\infty$ for any $s \in S_n^1$.

(c) Consider the error in best approximation to f from S_n^1 and the error in approximating f by its linear interpolant s_f. If one goes to zero, does the other go to zero?