Numerical Analysis
September 3, 2002

Do any 8 of the following 10 problems:

1. Consider a function $f(x) \in C^2(R)$ that satisfies the following properties:

   (a) There exists a unique root $\alpha \in [2,3]$.

   (b) For any real x, $f'(x) \geq 3$ and $0 \leq f''(x) \leq 5$.

   Using $x_0 = 5/2$, do we know that Newton's method will converge? If so, how many iterations are required to ensure 10^{-4} accuracy?

2. Let $x_0, \ldots, x_n$ be distinct real points. Let

$$P_n(x) = \sum_{j=0}^{n} c_j e^{jx}.$$

   For given data $y_0, \ldots, y_n$, show that there exists a unique choice of $c_0, \ldots, c_n$ such that

$$P_n(x_i) = y_i.$$

   (Hint: Reduce to an ordinary interpolation problem)

3. Suppose you want to solve numerically the ode

$$y' = f(x,y) \quad y(0) = y_0.$$

   (a) By applying Simpson's rule to the integral in the equation

$$y(x_{n+1}) = y(x_{n-1}) + \int_{x_{n-1}}^{x_{n+1}} f(s, y(s))ds, \quad x_n = nh, \quad i = 1, 2, \ldots$$

   derive the multistep Simpson's method for odes.

   (b) Find the order of the local truncation error for this method.

   (c) Determine if this method is stable or not.

4. (a) Find the linear least squares approximation to $f(x) = x^3$ on the interval $[0,2]$ by optimizing the least squares error over a and b in $r^*(x) = ax + b$.

(b) Find the first two orthonormal polynomials $\phi_0(x), \phi_1(x)$ on $[0, 2]$ (weight function $w(x) = 1$). Use these to find the linear least squares approx to $f(x) = x^3$, and check this result agrees with the previous problem.

5. (a) Let $\mathbf{A} \in \mathbb{C}^{m \times n}$ with $m \geq n$. Prove that $\mathbf{A}^*\mathbf{A}$ is nonsingular if and only if $\mathbf{A}$ has full rank.

   (b) Let the $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_\ell \in \mathbb{C}^m$ be linearly independent vectors. Form an $m \times \ell$ matrix $\mathbf{A}$ whose i-th column is $\mathbf{a}_i$. Prove that the matrix of the orthogonal projection onto span of $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_\ell$, is $\mathbf{A}(\mathbf{A}^*\mathbf{A})^{-1}\mathbf{A}^*$.

6. Let p and q are positive real numbers such that $1/p + 1/q = 1$, and let $\|\cdot\|_p$ for $1 \leq p \leq \infty$ denote the norm on $m \times n$ matrices induced by the ℓ^p-norm on vectors in $\mathbb{C}^m$ and $\mathbb{C}^n$.

   (a) For any matrix, show that $\|\mathbf{A}\|_p = \|\mathbf{A}^\mathsf{T}\|_q$ (Hint: use the Hölder inequality for vectors).

   (b) Let $\rho(\mathbf{M})$ denote the spectral radius of any square matrix $\mathbf{M}$. Prove that $\rho(\mathbf{M}) \leq \|\mathbf{M}\|$ for any norm $\|\cdot\|$ induced by a vector norm.

   (c) Show that $\|\mathbf{A}\|_2 \leq \sqrt{\|\mathbf{A}\|_p \|\mathbf{A}\|_q}$.

7. (a) Show that Jacobi's iteration applied to $\mathbf{Ax} = \mathbf{b}$ always converges when the matrix is row diagonally dominant.

   (b) Show that the Gauss-Seidel iteration applied to $\mathbf{Ax} = \mathbf{b}$ always converges when the matrix is row diagonally dominant.

8. If $\mathbf{A}$ is a square matrix, then $e^\mathbf{A}$ is the matrix obtained by forming the Taylor expansion of e^x and replacing x by $\mathbf{A}$. For any square matrix, show that $\det e^\mathbf{A}$ is the product of the exponential of each eigenvalue of $\mathbf{A}$.

9. If $\mathbf{A}$ and $\mathbf{B}$ are square invertible matrices and $\mathbf{u}$ and $\mathbf{v}$ are vectors with $\mathbf{A} = \mathbf{B} - \mathbf{uv}^\mathsf{T}$, obtain a formula for the scalar α in the following identity relating the inverses of $\mathbf{A}$ and $\mathbf{B}$:

$$\mathbf{A}^{-1} = \mathbf{B}^{-1} + \alpha \mathbf{B}^{-1}\mathbf{uv}^\mathsf{T}\mathbf{B}^{-1}$$

10. Let $\mathbf{A}$ be an $n \times n$ Hermitian positive definite matrix. Define

$$\|\mathbf{y}\|_A = (\mathbf{y}^* \mathbf{A} \mathbf{y})^{1/2}, \qquad \text{for all } \mathbf{y} \in \mathbb{C}^n.$$

Consider the following iterative method for solving $\mathbf{Ax} = \mathbf{b}$:

$$\mathbf{x}_{i+1} = \mathbf{x}_i + \alpha_i \mathbf{r}_i, \qquad \text{where } \mathbf{r}_i = \mathbf{b} - \mathbf{A}\mathbf{x}_i, \text{ and } \alpha_i = \frac{\mathbf{r}_i^* \mathbf{r}_i}{\mathbf{r}_i^* \mathbf{A} \mathbf{r}_i}. \tag{1}$$

Let error at the i-th step be $\mathbf{e}_i \equiv \mathbf{x} - \mathbf{x}_i$.

(a) Prove that

$$\|\mathbf{e}_{i+1}\|_A = \inf_{\alpha \in \mathbb{R}} \|\mathbf{e}_i - \alpha \mathbf{r}_i\|_A.$$

(b) Use Problem (10a) to prove the following convergence rate estimate:

$$\|\mathbf{e}_{i+1}\|_A \le \left(\frac{\kappa(\mathbf{A}) - 1}{\kappa(\mathbf{A}) + 1} \right) \|\mathbf{e}_i\|_A,$$

where $\kappa(\mathbf{A})$ denotes the spectral condition number of $\mathbf{A}$.