Numerical Analysis
Preliminary Exam
January 22, 2002

Do any 8 of the following 10 problems.

Part 1: Numerical Linear Algebra

1. (a) Prove: If $\mathbf{A} \in \Re^{\mathbf{n}\times\mathbf{n}}$ has a set of n linearly independent eigenvectors, then $\mathbf{A}$ can be diagonalized.
(b) Prove: If $\mathbf{A} \in \Re^{\mathbf{n}\times\mathbf{n}}$ is symmetric, then A has an orthogonal set of eigenvectors.
(c) Diagonalize the matrix
$$\mathbf{A} = \begin{pmatrix} 66 & 12 \\ 12 & 59 \end{pmatrix}.$$

2. (a) Is the matrix
$$\mathbf{A} = \begin{pmatrix} 5 & 1 \\ 0 & 3 \end{pmatrix}$$
diagonalizable? Why?
(b) Can the above matrix $\mathbf{A}$ be diagonalized by an orthogonal matrix $\mathbf{S}$?
(c) State a theorem or condition that ensures a matrix can be diagonalized by an orthogonal matrix.

3. (a) Give a careful statement of the singular value decomposition.
(b) Give a careful proof of the singular value decomposition.
(c) Compute the SVD for the matrix
$$\mathbf{A} = \begin{pmatrix} -6 & 17 \\ 18 & -1 \end{pmatrix}.$$

4. Let
$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
(a) Determine the orthogonal projection of $\mathbf{A}$ onto the column space of $\mathbf{A}$.
(b) Compute the QR Factorization of $\mathbf{A}$.

5. (a) Define: The operator 2-norm for a matrix $\mathbf{A}$ is any matrix in $\Re^{m\times n}$.
(b) Prove: If $\mathbf{A} \in \Re^{\mathbf{m}\times\mathbf{n}}$ and $\mathbf{B} \in \Re^{\mathbf{n}\times\mathbf{q}}$, then $||\mathbf{BA}||_2 \leq ||\mathbf{B}||_2 \cdot ||\mathbf{A}||_2$.
(c) Prove: If $\mathbf{A}$ is any matrix in $\Re^{n\times n}$ and $||\mathbf{A}||_2 < \mathbf{1}$, then $\lim_{n\to\infty} \mathbf{A}^{\mathbf{n}} = \mathbf{0}$.
(d) Prove: If $\mathbf{A}$ is a matrix in $\Re^{n\times n}$, then define $e^{\mathbf{A}}$ as a power series expansion. Show this series converges.
(e) Prove: If $\mathbf{A} \in \Re^{n\times n}$ is invertible, $\mathbf{b}$ and $\Delta\mathbf{b}$ are vectors in $\Re^n$, $\mathbf{x}$ and $\Delta\mathbf{x}$ are vectors in $\Re^n$ are solutions to $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{A}\Delta\mathbf{x} = \Delta\mathbf{b}$, respectively, then
$$\frac{||\Delta\mathbf{x}||_2}{||\mathbf{x}||_2} \leq \kappa_2(\mathbf{A})\frac{||\Delta\mathbf{b}||_2}{||\mathbf{b}||_2}.$$

Part II: Numerical Analysis

6. (a) State Newton's method (the algorithm) for solving $f(x) = 0$ where $f : R \to R$.
   (b) Assuming f is smooth and we have an initial guess sufficiently close to the root, state sufficient condition(s) for the method to converge quadratically.
   (c) Prove: Let $T(x) : [a, b] \to [a, b]$ be a function such that $T'''(x)$ is continuous for all $x \in [a, b]$, $T(p) = p, T'(p) = T''(p) = 0$. If $x_0 \in [a, b]$ is a point with the property that the sequence defined recursively by $x_{k+1} = T(x_k)$ converges to p, then the sequence $\{x_k\}_{k=1}^{\infty}$ converges cubically to p.

7. (a) Give a careful statement of the minimization property for clamped cubic splines.
   (b) Given a careful statement of the convergence theorem for clamped cubic splines.
   (c) Given a careful statement of the convergence theorem for the second derivatives for clamped cubic splines.
   (d) Outine a proof of the convergence theorem for clamped cubic splines.

8. (a) Given a set of data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$ in the plane, describe how to find the conic section of the form $Ax^2 + Bxy + Cy^2 + Dx + Ey + 1 = 0$ which best fits the data in the sense of least squares.
   (b) How do you determine whether or not the conic section is an ellipse, a hyperbola, or a parabola?

9. **Triple Recursion Formula** If $\{\phi_n(x)\}$ is an orthogonal family of polynomials on $[a, b]$, with respect to the weight function $w(x) \geq 0$, and $n \geq 1$, then show that $\phi_{n+1}(x) = (a_n x + b_n)\phi_n(x) - c_n\phi_{n-1}(x)$, for some constants a_n, b_n, and c_n. (Hint: Let $g(x) = \phi_{n+1}(x) - a_n x \phi_n(x)$, where a_n is a constant chosen so that $g(x)$ has degree n.)

10. (a) Give a careful statement of the contraction mapping theorem.
    (b) Prove the contraction mapping theorem.
    (c) Determine whether or not the function $T(x) = \frac{1}{4}\sin(3x + 2) + 12$ is a contraction.
    (d) Discuss the convergence rate ensured by the contraction mapping theorem. (ie Is it linear, quadratic, or cubic?)