Numerical Analysis
Preliminary Exam

May 18, 2000

Part 1: Numerical Linear Algebra

1. (a) State the conditions under which a square matrix A can be decomposed into a form $A = SDS^{-1}$, where D is diagonal.
   (b) Can
   $$A = \begin{pmatrix} 5 & 1 \\ 0 & 3 \end{pmatrix}$$
   be diagonalized in the sense of (a)? Why?

2. (a) State the procedure for factoring a matrix A into a form $A = LU$, where L is lower triangular, and U is upper triangular.
   (b) When can this type of decomposition be accomplished?
   (c) What is the computational advantage of having lower and upper triangular matrices. An explanation without specific computation counts is sufficient.

3. Assume that a symmetric matrix can be factored into a Cholesky decomposition $A = LL^t$, where L is lower triangular.
   (a) Give an exact numerical method for finding the specific entries $l_{i,j}$ of $L = \{l_{i,j}\}$, where $i = 1 : n$, and $j = 1 : n$?
   (b) What numerical problems would you anticipate with this method?

4. (a) Describe the method for factoring a matrix A into a form $A = QR$, where Q is orthogonal, and R is upper triangular.
   (b) When can such a decomposition be accomplished?
   (c) What numerical "tricks" can be used to make it more stable?

5. (a) State the Gershgorin Circle Theorem.
   (b) Prove the first assertion, i.e. that the eigenvalues must be within one of the circles.
   (c) Let
   $$A = \begin{pmatrix} 5 & .01 & .03 \\ .02 & 4 & .01 \\ .03 & -.01 & 4 \end{pmatrix}.$$
   What can you say, from the Gershgoring Circle Theorem, about the eigenvalues of A.

Part II: Numerical Analysis

6. Suppose that a sequence is defined by $x_{n+1} = g(x_n)$, where g is continuous, and $g([a,b]) \subset [a,b]$.
(a) Show that there is a fixed point $\alpha \in [a,b]$ such that $g(\alpha) = \alpha$.
(b) If in addition, $|g'(x)| \leq k < 1$ prove that this fixed point must be unique.
(c) Under the above assumptions show that the sequence must converge to the unique fixed point.

7. (a) Give an exact expression for a Lagrange polynomial of degree n, which would interpolate $n+1$ given data points, $f(x_k)$.
(b) Describe the differences between Hermite polynomials and Lagrange polynomials.
(c) Describe a procedure for using a limiting set of Lagrange polynomials, which interpolates a function at the points $x_0, x_0+h, x_1, x_1+h, x_2, x_2+h, \ldots$ to construct the Hermite polynomial at the points $x_0, x_1, x_2 \ldots$.

8. Given the Lagrange interpolation formula for $f(x)$,

$$f(x) = f(-h)\frac{x-h}{-2h} + f(h)\frac{x+h}{2h} + f''(\zeta(x))\frac{(x-h)(x+h)}{2!};$$

(a) Derive the trapezoid rule for $\int_{-h}^{h} f(x)dx$,
(b) Derive the error formula for the trapezoid rule, and
(c) Describe the idea behind Gaussian Quadrature, and explain why it is more accurate than "traditional methods" such as trapezoid, Simpson's, etc.

9. (a) **Triple Recursion Formula** Let $\{\phi_n(x)\}$ be an orthogonal family of polynomials on $[a,b]$, with weight function $w(x) \geq 0$. Then for $n \geq 1$, show that

$$\phi_{n+1}(x) = (a_n x + b_n)\phi_n(x) - c_n\phi_{n-1}(x),$$

for some constants a_n, b_n, and c_n. (Hint: Let $g(x) = \phi_{n+1}(x) - A_n\phi_n(x)$, where A_n is a constant chosen so that $g(x)$ has degree n.
(b) What can you say about the location of the zeros of a family of orthogonal polynomials on $[a,b]$?

10. (a) Describe Euler's method for solving the initial value problem $y' = f(t,y)$, $y(a) = y_0$.
(b) Suppose that $|f(t,x) - f(t,y)| \leq L|x-y|$, for a fixed constant L, and all t, x and y. Suppose further that all solutions $y(t)$ to the initial value problem described in (a) satisfy $|y''(t)| \leq M$, for some constant M. Then if w_i is the Euler solution to the initial value problem, prove that

$$|y_{i+1} - w_{i+1}| \leq (1+hL)|y_i - w_i| + \frac{h^2M}{2}.$$