Ph.D. Numerical Analysis Exam **May 1999**

Do any 8 of the following 10 problems.

1.

(1) Give a careful statement and proof of the singular value decomposition.
(2) Give a careful statement of the QR factorization theorem.
(3) Indicate how the Householder transformations are used in the proof of the QR factorization theorem.
(4) Describe the QR Iteration Algorithm with shift μ.
(5) Describe how the QR Iteration is used to compute the SVD.
(6) How can the singular values be used to compute the condition number of a matrix.

2.

Give a careful proof of the following
Theorem: If $p \geq 1$,

1. $\mathbf{A} \in \mathbf{R}^{n\times n}$ is invertible,
2. $\mathbf{b}$ and $\Delta\mathbf{b}$ are vectors in $\mathbf{R}^n$, and
3. $\mathbf{x}$ and $\Delta\mathbf{x}$ are vectors in $\mathbf{R}^n$ are solutions to $\mathbf{Ax} = \mathbf{b}$, and $\mathbf{A}\Delta\mathbf{x} = \Delta\mathbf{b}$, respectively,

then
$$\frac{\|\Delta\mathbf{x}\|_p}{\|\mathbf{x}\|_p} \leq \kappa_p(\mathbf{A})\frac{\|\Delta\mathbf{b}\|_p}{\|\mathbf{b}\|_p}.$$

3.

(1) Describe how Jacobi rotations can be used to diagonalize a real symmetric matrix.
(2) Give the error estimate for a single Jacobi rotation.
(3) Give the error estimates for repeated Jacobi rotations.

4.

(1) Write out the system of linear equations to be solved to find the periodic spline interpolation for the given data points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, where $x_0 < x_1 < \cdots < x_n$ and $y_n = y_0$.
(2) What method would recommend for solving the system of linear equations. Explain!!

5.

(1) Give a careful definition of the Householder transformation.
(2) Explain why the Householder transformation is preferred over the Gram-Schmidt orthogonalization process for creating an orthonormal basis. (Give an example if necessary.)
(3) Show how Householder transformations can be used to bidiagonalize any real matrix.

6.

(1) Give a careful statement of the Pythagorean Theorem property for clamped cubic splines.
(2) Give a careful statement of the integral minimization property for clamped cubic splines.
(3) If $s_n(x)$ denotes the clamped cubic spline interpolant for a function $f(x)$ at the knots $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$, then give a precise statement for the error between $s_n(x)$ and $f(x)$.
(4) Prove the error formula given in step (3) above.

7.

(1) If $p_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$, then show that $\int_{-1}^{1} x^m p_n(x) dx = 0$ for all $m < n$. (Hint: Think small values of m and n, induction, and parts.)
(2) What can you conclude about the relationship between the polynomials $p_n(x)$ and the Legendre polynomials?
(3) Explain the method of Gauss quadrature for the numerical approximation of the integral of a function.

8.

(1) Discuss the ideas behind the linear shooting method for approximating the solution of a linear two point boundary value problem. In particular, explain how the Runga-Kutta method can be used in the approximation.
(2) Describe how the Galerkin method can be used to solve a 2 point boundary value problem on the interval $[0, 1]$.
(3) Explain why splines have an advantage over $sin(kx)$ as choices of test functions for the Galerkin method.

9.

(1) Give a careful statement of the Contraction Mapping Theorem.
(2) Outline a proof of the Contraction Mapping Theorem.
(3) Indicated how Aitken's method can be used to accelerate convergence.

10.

(1) Give a careful statement of the error formula for the trapezoidal rule for approximating an integral.
(2) Give a careful proof of the error formula for the trapezoidal rule.