Ph.D. Numerical Analysis Exam August 1995

1. Consider the Householder matrix $H = I - 2ww^T$ where $w^T w = 1$.

(a) Show that H is orthogonal.

(b) Determine the eigenvalues of H.

(c) Given a vector x and an integer k, find a vector w such that $(Hx)_i = x_i$ for $i < k$ and $(Hx)_i = 0$ for $i > k$.

(d) Explain why one needs to be careful about the choice of sign in the formula for w.

2. Let A be a symmetric, positive definite matrix. Show that if A is LU factored, then the elements of U in absolute value are bounded by the largest diagonal element of A.

3. Give a careful statement and proof of Gershgorin's Theorem. Find upper and lower bounds for the set of eigenvalues for the matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

4. Let A be a nonsingular $n \times n$ matrix and $\{X_k\}$ a sequence of $n \times n$ matrices generated by the iteration:

$$X_{k+1} = X_k + X_k(I - AX_k).$$

Show that if X_0 is chosen such that the spectral radius of $I - AX_0$ is strictly less than 1, then $\{X_k\}$ converges to the inverse of A.

5. For a given partition $\{x_0, x_1, \cdots, x_n\}$ of an interval $[a, b]$ and function f defined on $[a, b]$

(a) State the divided difference formula for the polynomial $p_n(x)$ interpolating f from the data $(x_k, f(x_k))$, $k = 0, 1, \cdots, n$.

(b) Show that the error in the approximation in part (a) is given by

$$f(x) - p_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)f[x_0, x_1, \cdots, x_n, x]$$

(c) For $\{x_0, x_1, x_2\} = \{0, 1, 2\}$ and $f(x) = 1/(1+x^2)$, compute the divided difference formula for $p_2(x)$.

6. Let A be a real, positive definite $n \times n$ matrix and b be an $n \times 1$ vector.

(a) Derive the steepest descent algorithm for approximating the true solution x of $Ax = b$, by a sequence of iterates $\{x_k\}$.

(b) Define the real–valued function h on $\mathbf{R}^n$ by

$$h(x) = (Ax - b)^T A^{-1}(Ax - b)$$

and show that

$$h(x_{k+1}) \leq (1 - c(A)^{-1})^2 h(x_k)$$

where $c(A)$ is the condition number of the matrix A.

7. For an algorithm, define the terms inherent error, total effect of rounding, and numerically stable.

Carefully formulate a *numerically stable* algorithm for computing the positive solution of the quadratic equation $x^2 + 2px - q = 0$ where p and q are arbitrary positive numbers. Justify the numerical stability of your algorithm.

8. Give a careful proof that the DFT of a convolution of two finite sequences (not necessarily of the same length) is proportional to the product of the DFT's of the original sequences, suitably padded.

9. Given the N sampled values $f(2k\pi/N)$, $k = 0, 1, 2, \ldots, N-1$, of a twice continuously differentiable, 2π–periodic function f, outline a method of computing the Fourier coefficients of f which has the following properties

- makes use of the FFT routine
- produces an estimate which improves as $N \to \infty$.

Estimate the accuracy of your method.

If possible, state an explicit formula for its implementation.