Ph.D. Numerical Analysis Exam — May 1995

1. Consider the linear system $Ax = b$.

(a) Suppose the columns of A are linearly independent. Derive the normal equation for the x that minimizes the 2–norm of the residual $r = b - Ax$.

(b) Suppose that the rows of A are linearly independent. Derive the normal equation for the solution to $Ax = b$ that has the smallest 2–norm.

(c) Using the singular value decomposition, obtain a formula for the least squares solution to $Ax = b$. That is, if M denotes the set of all x for which the 2–norm of the residual is minimal, your formula should give the point in M with smallest 2–norm.

2. Suppose the columns of A are independent. Given the QR factorization of A, give a formula for the distance from some given vector b to the space spanned by the columns of A.

3. Verify the Woodbury formula

$$(B - UV^T)^{-1} = B^{-1} + B^{-1}U(I - V^TB^{-1}U)^{-1}V^TB^{-1}.$$

4. Explain how the singular value decomposition can be used to determine the rank of a matrix in the presence of rounding errors.

5. Derive a formula for the 2–norm of a matrix in terms of its singular values.

6. Let f be an $(n+1)$–times continuously differentiable function on the interval $[a,b]$, and let $a \le x_0 < x_1 < \dots < x_n \le b$. Let P_n be the unique polynomial interpolant of degree at most n, interpolating the data $(x_k, f(x_k))$, $k = 0, 1, \dots, n$.

Prove that there exists a point $\xi \in [a,b]$ such that

$$f(x) - P_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\prod_{k=0}^{n}(x - x_k)$$

for all $x \in [a,b]$.

7. Let f be a twice continuously differentiable function on the interval $[a,b]$, and $x_0, x_1, \dots, x_N$ be an equi–spaced partition of $[a,b]$, with step size h and $x_0 = a$. Assume that only the values $f(x_k)$, $k = 0, 1, \dots, N$ are known.

(a) State the trapezoidal rule $T(f,N)$ to approximate $\int_a^b f(x)dx$.

(b) Prove that

$$\int_a^b f(x)dx - T(f,N) = -\frac{(b-a)^3}{12N^2}f''(\xi)$$

for some point $\xi \in (a,b)$.

8. Let f be a twice continuously differentiable, 2π-periodic, function on $\mathbb{R}$ and consider the problem of computing the Fourier coefficients

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{-inx} f(x)dx \tag{8.1}$$

by using the values $f_k = f(2k\pi/N)$ for $k = 0, 1, \ldots, N-1$.

(a) Define the discrete Fourier transform $\{\hat{f}_n\}$ of the sequence $\{f_k\}$.

(b) By applying the trapezoidal rule to the integral in (8.1), show that c_n can be approximated by $\hat{f}_n$.

(c) By performing an analysis of the error in the approximation in (b), show that $\hat{f}_n$ is a poor approximation of c_n. Hint: Investigate behavior of error as $N \to \infty$.

(d) Outline a method of obtaining a better approximation $\tilde{c}_n$ of c_n. (You do not need to actually find a formula for this $\tilde{c}_n$.) For the $\tilde{c}_n$ obtained by your method, define

$$\|\tilde{c} - c\|_\infty = \max\{|\tilde{c}_n - c_n|,\ n = 1, 2, \ldots\}$$

Find the order of convergence of $\|\tilde{c} - c\|_\infty$ as $N \to \infty$.

9. Give a careful statement of the contraction mapping theorem for functions defined on subsets of $\mathbf{R}^n$ into $\mathbf{R}^n$.

(a) Since it is usually not easy to demonstrate the contraction property in particular examples, state a stronger condition which implies the contraction property.

(b) Discuss the relative merits of determining the fixed points of a function $\varphi : \mathbf{R} \to \mathbf{R}$ by the iterations:

(FP) $$x_{k+1} = \varphi(x_k)$$

(DFP) $$x_{k+1} = \varphi(\varphi(x_k))$$

(STEF) $$x_{k+1} = x_k - \frac{(\varphi(x_k) - x_k)^2}{\varphi(\varphi(x_k)) - 2\varphi(x_k) + x_k}$$

10. Let A be a real, positive definite $n \times n$ matrix and b be an $n \times 1$ vector.

(a) Derive the steepest descent algorithm for approximating the true solution x of $Ax = b$, by a sequence of iterates $\{x_k\}$.

(b) Define the real-valued function h on $\mathbf{R}^n$ by

$$h(x) = (Ax - b)^T A^{-1}(Ax - b)$$

and show that

$$h(x_{k+1}) \leq (1 - c(A)^{-1})^2 h(x_k)$$

where $c(A)$ is the condition number of the matrix A.