PhD Functional Analysis Examination
September 2011

Do any SIX problems. Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

1. Let X be a vector space and $\mathcal{F}$ a vector space of linear functionals on X that separates points.

   a) Write down a base for the weak topology on X induced by $\mathcal{F}$.

   b) Prove that if a linear functional φ on X is continuous in this topology, then $\varphi \in \mathcal{F}$.

2. Let X be a normed vector space. Suppose (x_n) is a sequence in X, converging weakly to $x \in X$. (That is, $\varphi(x_n) \to \varphi(x)$ for every norm-continuous linear functional φ.) Prove that there is a sequence y_n, where each y_n is a convex combination of finitely many of the x_n, such that $\lim \|y_n - x\| = 0$.

3. Let T be a bounded linear operator on a Hilbert space H. Prove that if both T and T^* are bounded below, then T is invertible. (Recall that T bounded below means that there is a constant $c > 0$ so that $\|Tx\| \geq c\|x\|$ for all $x \in H$.)

4. Let T be a bounded linear operator on Hilbert space. Prove that there is a positive operator P and a partial isometry U such that $\ker U = \ker T$ and $T = UP$ (that is, T has a *polar decomposition*).

5. Let T be a Hilbert space operator with $\|T\| \leq 1$.

   a) Prove that $I - T^*T$ and $I - TT^*$ are positive operators.

   b) Prove that
   $$T(I - T^*T)^{1/2} = (I - TT^*)^{1/2}T.$$
   (Hint: approximate $\sqrt{1-x}$ by polynomials $p_n(x)$ and use the spectral theorem.)

6. Let (λ_n) be a bounded sequence of complex numbers. Consider the *weighted shift operator* S defined on $\ell^2(\mathbb{N})$ by
   $$S(x_0, x_1, x_2, \dots) = (0, \lambda_0 x_0, \lambda_1 x_1, \lambda_2 x_2, \dots).$$
   Characterize those sequences (λ_n) for which S is compact.

7. Let A be a Banach algebra (over $\mathbb{C}$) and $a \in A$.

   a) Define the *spectrum* of a.

   b) Show that if $\lambda \in \mathbb{C}$ and $|\lambda| > \|a\|$, then λ does not lie in the spectrum of a.