PHD ERGODIC THEORY AND DYNAMICAL SYSTEMS EXAM, SUMMER 2013

1. Suppose that X is a compact metric space, and $f : X \to X$ is continuous. Suppose that X is a minimal set for f. Prove that every point of X is almost periodic.

2. Let $f : [0, 1] \to [0, 1]$ be continuous. Suppose that f has a periodic point of odd period larger than one. Prove that the topological entropy of f is at least $\frac{\log(2)}{2}$.

3. Describe the construction of the homeomorphism of the sphere to itself known as the Horseshoe Map (of S. Smale). Prove that the set of periodic points is dense in the nonwandering set for this map.

4. Let X be a topological space, and let $f : X \to X$ be a Borel measurable function. Suppose that there exists a unique f-invariant Borel probability measure, μ. Prove that μ is ergodic.

5. Let $T_1 : [0, 1] \to [0, 1]$ denote the full tent map, let $\mathcal{B}_1$ denote the collection of Borel sets, and let μ_1 denote Lebesgue measure . Recall that T_1 is measure preserving with respect to the probability space $([0, 1], \mathcal{B}_1, \mu_1)$. Let $T_2 : \Sigma_2^+ \to \Sigma_2^+$ denote the full one-sided shift map on two symbols, let $\mathcal{B}_2$ denote the σ-algebra generated by the cylinder sets, and let μ_2 denote the $(\frac{1}{2}, \frac{1}{2})$ product measure. Recall that T_2 is measure preserving with respect to the probability space $(\Sigma_2^+, \mathcal{B}_2, \mu_2)$.

Are T_1 and T_2 isomorphic? Prove your answer.

6. Suppose that X is a compact metric space, and $f : X \to X$ is continuous. Prove that there exists an f-invariant Borel probability measure.

7. Let $(X, \mathcal{B}, \mu)$ be a probability space. Suppose that $T : X \to X$ is measure-preserving. Prove that the following are equivalent.

(a) T is ergodic

(b) For every f in $L^1(\mu)$, the sequence of real numbers

$$\| \int f d\mu - \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \|_1$$

approaches 0 as $n \to \infty$.

(c) For every bounded f in $L^1(\mu)$, the sequence of real numbers

$$\| \int f d\mu - \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k \|_1$$

approaches 0 as $n \to \infty$.

8. Let $f : [0, 2] \to [0, 2]$ be the piecewise linear continuous map given by $f(x) = x + 1$, if $0 \leq x \leq 1$ and $f(x) = 4 - 2x$, if $1 \leq x \leq 2$. Determine whether each of the following statements is true or false. Prove your conclusions.

(a) Lebesgue measure is an invariant measure for f.

(b) There exist infinitely many (distinct) invariant ergodic measures for f.

(c) The map f has a periodic point with period 837.

(d) The map f is forward orbit topologically transitive.

(e) Each point of $[0, 2]$ is nonwandering under$f$.