Dynamical Systems and Ergodic Theory PhD Exam – May 2012

Instructions: Do all problems, each on a separate page.

Notation:

- The statement that $(X, \mathcal{B}, \mu, f)$ is a *mpt* (measure preserving transformation) means that X is a set with the probability measure μ defined on the σ-algebra $\mathcal{B}$ and f is a bijective, bi-measurable map $f : X \to X$ which preserves the measure μ.
- The statement that $(X, d, \mathcal{B}, \mu, f)$ is a *mph* (measure preserving homeomorphism) means that $(X, \mathcal{B}, \mu, f)$ is a mpt and in addition, (X, d) is a compact metric space, f is a homeomorphism, and $\mathcal{B}$ is the Borel σ-algebra.

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1. Let (X, d) be a compact metric space and $f : X \to X$ a homeomorphism.
   (a) Define the *recurrent set* $\Lambda(f)$ and the *nonwandering set* $\Omega(f)$ and show they are both compact, invariant sets.

   (b) Show that $\Lambda(f) \subset \Omega(f)$ and give an example where the inclusion is proper.

   (c) Define the *omega-limit set* $\omega(x, f)$ of a point $x \in X$ and show it is a compact, invariant set.

   (d) Prove or disprove: For every $x \in X$, that $\omega(x, f) \subset \Lambda(f)$.

2. Let (X, d) be a compact metric space and $f : X \to X$ a homeomorphism.
   (a) Show that there is always an f-invariant subset $Z \subset X$ so that f restricted to Z is minimal.

   (b) Show that there is always a measure μ so that $(X, d, \mathcal{B}, \mu, f)$ is a mph.

3. Assume $(X, d, \mathcal{B}, \mu, f)$ is a mph.
   (a) Define the support, $\text{supp}(\mu)$, of the measure μ and show it is a compact, invariant set.

   (b) Show that almost every point in $\text{supp}(\mu)$ is recurrent.

   (c) Show that every point in $\text{supp}(\mu)$ is nonwandering.

   (d) Show that $\mu(\Omega(f)) = 1$.

   (e) If $(X, d, \mathcal{B}, \mu, f)$ is ergodic, show that f is full orbit transitive on $\text{supp}(\mu)$.

   (f) Show that $\text{supp}(\mu) \subset \Lambda(f)$, where $\Lambda(f)$ is the recurrent set of f

4. State carefully with complete hypothesis.
   (a) Birkhof's pointwise ergodic theorem

   (b) Von Neumann's L^2-mean ergodic theorem

5. Let

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

   (a) Define the one-sided subshift of finite type Σ_A^+ determined by A and the shift map $\sigma : \Sigma_A^+ \to \Sigma_A^+$.

   (b) Define one of the standard metrics on Σ_A^+.

   (c) Show that periodic points of σ are dense in Σ_A^+.

   (d) Show that (Σ_A^+, σ) is forward orbit transitive.

   (e) Let N_k be the number of fixed points of fixed points of σ^k in Σ_A^+. Compute explicitly

$$\lim_{k\to\infty} \frac{\log(N_k)}{k}$$

   and justify your answer completely.

6. Fix a p with $0 < p < 1$.
   (a) Define the Bernoulli measure μ_p on the two-sided shift on two symbols Σ_2.

   (b) Show that $(\Sigma_2, \mathcal{B}, \sigma, \mu_p)$ is strong mixing (you don't have to prove it is a mpt, you may assume that is true).

   (c) Define $\alpha : \Sigma_2 \to \mathbb{R}$ by

$$\alpha(\dots s_{-1}s_0s_1s_2\dots) = 2s_{-1} - s_0 + s_2^2.$$

   Compute the following limit for μ_p-almost every sequence in $\underline{s} \in \Sigma_2$:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} \alpha(\sigma^i(\underline{s})).$$

   and justify your answer completely.

7. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x, y) = (3x, (1/2)y)$.
   (a) Show that the origin is an unstable fixed point for f.

   (b) Find an invariant, Borel probability measure for f (Hint: don't try to hard).

   (c) Show that your invariant measure in (b) is the only invariant, Borel probability measure.