Differential Geometry
PhD Examination
May 2007

Attempt SIX problems. Write solutions in a neat and logical fashion, giving complete reasons for all steps and stating carefully any substantial theorems used.

1. Decide which of the following are submanifolds of $\mathbb{R}^3$:
   (a) $A = \{(x, y, z) : z^2 = x^2 + y^2 - 1\}$;
   (b) $B = \{(x, y, z) : z^2 = x^2 + y^2 + 1\}$;
   (c) $C = \{(x, y, z) : z^2 = x^2 + y^2\}$.

2. Defining the terms involved, prove the *Cartan identity*

$$L_\zeta = (\zeta \lrcorner) \circ \mathrm{d} + \mathrm{d} \circ (\zeta \lrcorner)$$

for the Lie derivative of differential forms along the vector field ζ.

3. Let $\theta \in \Omega^1(M)$ be nonsingular (that is, nowhere zero) with $\theta \wedge \mathrm{d}\theta = 0$. Briefly explaining why it is possible, choose $\alpha \in \Omega^1(M)$ such that $\mathrm{d}\theta = \theta \wedge \alpha$ and choose $\beta \in \Omega^1(M)$ such that $\mathrm{d}\alpha = \theta \wedge \beta$.
   (a) Prove that the three-form $\alpha \wedge \mathrm{d}\alpha$ on M is closed.
   (b) Prove that if also $\mathrm{d}\theta = \theta \wedge \alpha'$ then $\alpha' \wedge \mathrm{d}\alpha' - \alpha \wedge \mathrm{d}\alpha$ is exact.

4. Let $k \in \mathbb{R}$ and define $\omega_m \in \Omega^m(\mathbb{R}^{m+1} - \{0\})$ by

$$(x_0^2 + \cdots + x_m^2)^k \, \omega_m = \sum_{j=0}^{m} (-1)^j \, x_j \mathrm{d}x_0 \wedge \cdots \wedge \widehat{\mathrm{d}x_j} \wedge \cdots \wedge \mathrm{d}x_m$$

where the circumflex $\widehat{\cdots}$ over a term indicates that it is omitted.
   (a) Determine the value(s) of k (if any) for which the m-form ω_m is closed.
   (b) Determine the value(s) of k (if any) for which the 2-form ω_2 is exact.

5. Define the *Poisson bracket* between smooth functions on a symplectic manifold, both in invariant form and in terms of (local) Darboux coordinates.

   Let $\omega = \sum_{j=1}^{3} \mathrm{d}p_j \wedge \mathrm{d}q_j$ be the standard symplectic form on $\mathbb{R}^6$ and define

$$L_1 = q_2 p_3 - q_3 p_2, \quad L_2 = q_3 p_1 - q_1 p_3, \quad L_3 = q_1 p_2 - q_2 p_1.$$

Explaining the terms, prove that if L_1 and L_2 are constants of the motion for a given Hamiltonian function on $(\mathbb{R}^6, \omega)$ then so is L_3.

6. Let G be both a group and a smooth manifold; suppose that the group operation $p : G \times G \to G$ is smooth.

By considering the function

$$f : G \times G \to G \times G : (x, y) \mapsto (x, p(x, y))$$

or otherwise, prove that the inversion map $G \to G : g \mapsto g^{-1}$ is smooth.

[It may be assumed that the tangent map of p at (a, b) is given by

$$\xi \in T_a G, \eta \in T_b G \Rightarrow p_*(\xi, \eta) = \lambda^a_* \eta + \rho^b_* \xi$$

where $\lambda^a : G \to G \times G : y \mapsto (a, y)$ and $\rho^b : G \to G \times G : x \mapsto (x, b)$.]

7. Let $\sigma : G \times M \to M : (g, x) \mapsto \sigma_g(x)$ be a smooth (left) action of the Lie group G on the smooth manifold M. Explain how the induced map $\dot{\sigma} : \mathfrak{g} \to \mathrm{Vec}\, M$ is defined. For $\xi \in \mathfrak{g}$ and $x \in M$ define

$$\gamma : \mathbb{R} \to M : t \mapsto \sigma_{\exp t\xi}(x).$$

Show that the tangent vector to this curve at time t is given by

$$\dot{\gamma}(t) = (\sigma_{\exp t\xi})_* \dot{\gamma}(0).$$

Hence show that if the action σ is free and $\dot{\sigma}(\xi)_x = 0$ for some $x \in M$ then $\xi = 0$.