Combinatorics Exam

August 2017

1. Let a_n denote the number of strings of the letters A, B, C, and D such that the letter A appears an odd number of times.

   (a) Find a closed formula for a_n.

   (b) Find the exponential generating function for the sequence $\{a_n\}$.
   (using no summation signs)

2. The set $[n] \times [n]$ is partially ordered by the relation $(a, b) \preceq (c, d)$ which holds when $a \leq c$ and $b \leq d$. Find, with proof, the length of a maximum chain and the length of a maximum antichain.

   What is the size of a minimum chain decomposition of this poset. (Recall that a *chain decomposition* of a poset is a partition of its elements into disjoint chains.)

3. (a) Prove that, for a simple graph G with at least 5 vertices, at least one of G or its complement has a cycle.

   (b) Suppose that you color the edges of K_n using 2 colors. Show that there exists a monochromatic spanning tree.

4. A *parity check matrix* H of a binary linear code C can be defined as a generator matrix of the dual code $C^{\perp}$.

   Show that C is the null space of the transpose H^T, where multiplication by H^T is on the right, i.e., cH^T.

   Show thet the minimum distance of the code equals the cardinality of a minumum dependent set of columns of H.

   If a generator matrix of a binary linear code C is

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix},$$

   then show that $(c_1, c_2, c_3, c_4, c_5)$ is a codeword if and only if (modulo 2)

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + c_4 &= 0 \\ c_1 + c_5 &= 0. \end{aligned}$$

   What is the minimum distance of this code?

5. Let $g(n)$ be the number of permutations of length n in which each cycle is of even length. Find the exponential generating function of the sequence $g(n)$, where $n = 0, 1, 2, \cdots$.

6. Let t_n be the total number of cycles of all permutations of length n. So $t_1 = 1$, $t_2 = 3$, and $t_3 = 11$. Find an explicit formula for the numbers t_n. Your answer can contain one summation sign.

7. Prove that the language $\{\mathtt{a}^{n^2} : n \in \mathbb{N}\}$ is not regular.

8. Let β be a permutation of length k. Prove that there are precisely $k^2 + 1$ permutations of length $k + 1$ containing β.