Combinatorics Exam May 2011

1. Let $f(n)$ be the number of permutations of length n that have no cycle that is shorter than three.

   (a) Find the exponential generating function of the sequence $f(n)$. You can assume that $f(0) = 1$.

   (b) Find $\lim_{n \to \infty} f(n)/n!$.

2. We divide a group of n people into subgroups A, B, and C, and ask each subgroup to form a line. We also require that A have an odd number of people, and that B have an even number of people. How many ways are there to do this?

3. Let A be the graph obtained from K_n by deleting an edge. Find a formula for the number of spanning trees of A.

4. Let $X_n(p)$ be the number of cycles of the n-permutation p. Compute $VAR(p)$.

5. Show that a planar graph for which every face has an even number of edges must be bipartite.

6. Recall that a tournament is a complete directed graph. Prove that there exists a tournament on eight vertices that contains at least 316 Hamiltonian paths.

7. For which values of m and n is the complete bipartite graph $K_{m,n}$ planar, for which values of m and n is it Hamiltonian, and for which values of m and n is it Eulerian?

8. Prove in a finite poset P, the number of elements in the longest chain is equal to the smallest number k so that P can be decomposed into the union of k antichains.