COMBINATORICS EXAM

1. (a) Prove that

$$\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

(b) Prove that the number of solutions of $x_1 + \cdots + x_k \leq n$ in positive integers is $\binom{n}{k}$.

2. (a) Prove that the number of Hamiltonian cycles in the complete bipartite graph $K_{n,n}$ is $\frac{1}{2}(n-1)!n!$.

(b) Let e be an edge of the complete graph K_n. Use Cayley's Theorem to find the number of spanning trees in $K_n - e$.

3. Recall that the *average degree* of a graph G of order n is

$$av(G) = \frac{\sum_{v \in V} deg(v)}{n}.$$

(a) Prove that

$$\chi(G) \leq 1 + \Delta(G),$$

where χ denotes the chromatic number and Δ the maximum degree.

(b) For an arbitrary graph G, prove or give a counterexample to the statement

$$\chi(G) \leq 1 + av(G).$$

4. Let C be a binary code and let $\overline{C}$ be the code obtained from C by adding an even parity check. Find the minimum distance $d(\overline{C})$ in terms of $d(C)$.

5. Prove that balanced incomplete block designs with the following parameters do not exist:

$$(12, 8, 6, 4, 3), \qquad (22, 22, 7, 7, 2).$$

6. Find the unique sequence $\{a_n\}$ with

$$\sum_{k=0}^{n} a_k \, a_{n-k} = 1.$$

7. Use inclusion-exclusion to find a formula for the number of surjective functions $f : X \to Y$ with $|X| = n$ and $|Y| = m$.

8. Let $f(n, k)$ be the number of k-subsets of $\{1, 2, \ldots, n\}$ that do not contain a pair of consecutive integers.

(a) Find a recurrence for $f(n, k)$ by considering the k-subsets that do, and do not, contain 1.

(b) Use part (a) to show that $f(n, k) = \binom{n-k+1}{k}$.