Combinatorics

Aug 2005 PhD Exam

Show your work.

1a. There are $r$ black and $n-r$ white balls in an urn. They are removed one at a time without replacement. What is the probability that exactly $k$ drawings are required to get a white ball?

b. Use your result to conclude that

$$\sum_{k=1}^{r} \frac{(r)_k}{(n-1)_k} = \cancel{\frac{r}{n-2}}. \quad = \frac{n}{nr}$$

2. Each of $n$ people is to be mailed an envelope containing a letter and an bill. How many ways $Q_n$ are there of placing the $n$ letters and $n$ bills into $n$ addressed envolopes so that no envelope contains both the correct letter and bill?

3. Let $P_n$ be the total number of $k$-permutations of $n$ for various $k$, that is,

$$P_n = \sum_{k=0}^{n} (n)_k, \; n = 0, 1, \ldots.$$

Show that

$$P(t) = \sum_{n=0}^{\infty} P_n \frac{t^n}{n!} = (1-t)^{-1} e^t$$

and use this to show that

$$P_n = nP_{n-1} + 1, \; n = 1, 2, \ldots, P_0 = 1.$$

4. Define the binary Hamming code $H(r)$ of length $2^r - 1$. Show that $H(r)$ is an exactly single error correcting code and that $H(r)$ is a perfect code. Determine the number of codewords of weight 3 in $H(r)$.

5. Show that the number of partitions of a number $n$ into exactly $m$ parts is equal to the number of partitions of $n-m$ into no more than $m$ parts.

6. An order on the set of ordered pairs of non-negative integers is defined by $(a_1, a_2) \leq (b_1, b_2)$ if $a_i \leq b_i$ for $i = 1, 2$. Find the Mobius function of this poset.

7. Determine for which values of $m$ and $n$ the complete bipartite graph $K_{mn}$ is (a) planar, (b) Eulerian, (c) Hamiltonian.

Answer the same three questions for the $n$-cube. (Recall that the n-cube $Q_n$ is defined as the graph whose vertices are the set of all binary sequences of length $n$, where two vertices are adjacent if the corresponding sequences differ by exactly one digit.)

8. Let $G$ be a triangle-free graph with $n$ vertices, minimum degree $k$ and girth $g$. Prove that $g \leq 2n/k$.

Hint: For an appropriate cycle $C$ count the number of edges from $C$ to $G \setminus C$ in two ways.