Combinatorics
September 2002

PH.D. Exam

SHOW ALL WORK TO RECEIVE CREDIT!!

1. All n soldiers of a military squadron stand in a line. The officer in charge splits the line at several places, forming smaller (nonempty) units. Then he chooses a (possibly empty) subset of the newly formed units for night duty. In how many different ways can he do this?

2. All n soldiers of a military squadron stand in a line. The officer in charge splits the line at several places, forming smaller (nonempty) units. Then he names one person in each unit to be the commander of that unit. Let h_n be the number of ways he can do this, and let $H(x)$ be the ordinary generating function of the numbers h_n. Find $H(x)$.

3. Let G be a simple graph, and let A be the adjacency matrix of G. Decide whether the following statements are true or false.

   a. A has only real eigenvalues.

   b. The sum of the eigenvalues of A is 0.

   c. The determinant of A is always positive.

4. Prove that the number of ways to partition a convex $n+1$-gon into triangles and one quadrilateral by noncrossing diagonals is $\binom{2n-3}{n-3}$.

5. Decide whether B_n, Π_n, and NC_n are distributive lattices. (The Boolean algebra, the partition lattice, and the lattice of noncrossing partitions).

6. Let $f_k(n)$ be the number of permutations of length n having k valleys. Is it true that $f_k(n)$ is a p-recursive function of n, no matter what k is?

7. How many permutations of length 8 have descent set $\{1, 4, 6\}$?

8. Let $p = p_1 p_2 \cdots p_n$ be an n-permutation, and assume $n \geq 3$. We say that i is an *excedance* of p if $p_i > i$. Find the number of n-permutations whose set of excedances is $\{n-2, n-1\}$.

9. Let D be the partially ordered set of positive integers ordered by divisibility. Find a formula for $\mu(1, x)$.

10. Find an explicit formula for the numbers a_n if $a_{n+1} = (n+1)a_n + 2(n+1)!$ if $n \geq 1$, and $a_0 = 0$.